AIR FORCE REPORT NO.
SSD-TR-67-115

AEROSPACE REPORT NO.
TR-1001(9990)-2

Collected Results on Numerical Research

APRIL 1967

Prepared by T. R. PARKIN, and L. J. LANDER
Computation and Data Processing Center
Electronics Division
El Segundo Technical Operations
AEROSPACE CORPORATION

Prepared for COMMANDER SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
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Contract No. AF 04(695)-1001

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FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract No. AF 04(695)-1001.

This report, which documents research carried out from September 1965 through February 1967, was submitted on 2 June 1967 to Capt. Ronald J. Starbuck (SSTRT) for review and approval.

The authors acknowledge with gratitude the contributions of Dr. J. L. Selfridge who co-authored Section V, and the assistance of Miss Pauline Parkin who prepared the checking program mentioned in Section II.

Approved

D. R. S. McColl, General Manager
Electronics Division

Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

Ronald J. Starbuck, Capt., USAF
Chief, Space Environment and
Electronics Branch

ABSTRACT

A compilation of results from research in number theory involving the use of a digital computer is presented. The research is related to characteristics of prime numbers, equal sums of powers of integers, differences of powers of integers, equal sums of fifth powers, and the classic problem f finding two equal sums of two biquadrates.

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I. INTRODUCTION

This report describes research in the theory of numbers that involved the use of a digital computer.

Sections II and III concern characteristics of prime numbers and Sections IV and V problems in equal sums of powers of integers. In Section VI, the differences of powers of integers are discussed, while in VII and VIII two equal sums of fifth powers are considered. Sections IX and X provide the results of an investigation of the classic problem of finding two equal sums of two biquadrates.

The various sections of the report have been grouped for convenience. Although the sections are related in subject, each one is self-contained.

II. ON FIRST APPEARANCE OF PRIME DIFFERENCES

A. DISCUSSION

There has long been interest in strings of consecutive composite numbers appearing among the natural numbers. Most elementary texts on number theory include a discussion of how arbitrarily large gaps between consecutive primes can be constructed, for example [1]. Such constructive techniques lead to rather large numbers, however, and lower occurrences have been studied [2], [3] to gain insight into the subject.

In 1961, Gruenberger and Armerding examined the first six million primes (up to P = 104, 395, 289) [4] on a computer and produced certain statistics covering these primes [5]. They tabulated the primes forming the lower boundary for the first appearance of prime differences of prescribed lengths, where all intervening numbers are composite, up to the limit of the primes list. The largest difference found between two consecutive primes was 220, and the smallest difference whose first appearance was not found was 186.

An algorithm for direct search for prime-differences (usable on a computer of limited storage capacity) proceeds as follows:

a) Start at a known prime, say P_a, below which all differences of interest are known.

b) Form $P_a + D$, where D is the smallest difference whose first appearance is unknown.

c) From the point $P_a + D$, test the successively smaller numbers for primality by trial division or other technique until a prime P_b is found.

d) If $P_b > P_a$, replace P_a by P_b and repeat the algorithm.

e) If $P_b = P_a$, start testing at $P_a + D$, and proceed to successively larger numbers until a prime P_c is reached. $P_c - P_a$ is then a difference $\geq$ D between successive primes, and is recorded, unless such a difference has already occurred.

f) Update D, if necessary, to the next larger difference whose first appearance is unknown; replace P_a by P_c, and repeat the algorithm.

A computer program for the CDC 3200 was written to implement this algorithm, and Table I through the range $0 < P < 1.46 \times 10^9$ represents the data obtained from this program.

The algorithm itself guarantees that no difference of interest (i.e., $\geq$ smallest difference whose first appearance is unknown) will escape notice, while a separate check was run on the data in Table I. This check took the form of another computer program which read the Table I data as input, established the primality of P_a and P_b by testing for divisibility by primes up to the square root of P_a or P_b, and explicitly exhibited all the prime factors of each odd number between the two primes. Thus, the differences listed are verified to be exactly as long as stated. Since all previous results in [5] were exactly duplicated (items of Table I for $D \leq 184$ and $D = 196, 198, 210, 220$), the data may be regarded as accurate.

The primality testing process was designed to operate without using an extensive table of primes, while, at the same time, being made as rapid as practicable. First, the numbers to be tested were required to be prime to 210. Since a sequence of consecutive numbers was being tested, a single division by 210 followed by a table lookup in a table of 210 positions sufficed to exclude all numbers not prime to 210. Each entry of the table actually pointed to the next eligible number to be tested. Secondly, division by a few small primes was used. Since the ranges of interest quickly exceeded the single precision word length of the computer (24 bits), the 48-bit hardware arithmetic of the machine was used. However, in order to avoid double precision division as long as possible, the numbers being tested were reduced to single precision by subtraction of self-adjusting multiples of groups of small primes prior to division by those primes. The final step was division by all odd numbers prime to 6 (by alternately adding 2, then 4, to an appropriate starting prime) and less than the square root of the number tested, in order to verify the primality of the end points for the algorithm, and to determine that

Table I. First Appearance of Prime Differences

D	PA	PB	(LOG PB)/SQRT(D-1)
2*	3	5	1,609
4*	7	11	1,384
6*	23	29	1,506
8*	89	97	1,729
10	139	149	1,668
12	199	211	1,614
14*	113	127	1,344
16	1831	1847	1,942
18*	523	541	1,526
20*	887	907	1,562
22*	1129	1151	1,538
24	1669	1693	1,550
26	2477	2503	1,565
28	2971	2999	1,541
30	4297	4327	1,555
32	5591	5623	1,551
34*	1327	1361	1,256
36*	9551	9587	1,550
38	30593	30631	1,698
40	19333	19373	1,581
42	16141	16183	1,514
44*	15683	15727	1,474
46	81463	81509	1,686
48	28229	28277	1,495
50	31907	31957	1,482
52*	19609	19661	1,384
54	35617	35671	1,440
56	82073	82129	1,526
58	44293	44351	1,417
60	43331	43391	1,390
62	34061	34123	1,336
64	89689	89753	1,437
66	162143	162209	1,488
68	134513	134581	1,443
70	173359	173429	1,452

Table I. First Appearance of Prime Differences (Continued)

D	PA	PB	(LOG PB)/SQRT(D-1)
72*	31397	31469	1.229
74	404597	404671	1.511
76	212701	212777	1.417
78	188029	188107	1.384
80	542603	542683	1.486
82	265621	265703	1.388
84	461717	461801	1.432
86*	155921	156007	1.297
88	544279	544367	1.416
90	404851	404941	1.369
92	927869	927961	1.440
94	1100977	1101071	1.443
96*	360653	360749	1.313
98	604073	604171	1.352
100	396733	396833	1.296
102	1444309	1444411	1.411
104	1388483	1388587	1.394
106	1098847	1098953	1.357
108	2238823	2238931	1.414
110	1468277	1468387	1.360
112*	370261	370373	1.217
114*	492113	492227	1.233
116	5845193	5845309	1.453
118*	1349533	1349651	1.305
120	1895359	1895479	1.325
122	3117299	3117421	1.359
124	6752623	6752747	1.418
126	1671781	1671907	1.282
128	3851459	3851587	1.346
130	5518687	5518817	1.367
132*	1357201	1357333	1.234
134	6958667	6958801	1.366
136	6371401	6371537	1.348
138	3826019	3826157	1.295
140	7621259	7621399	1.344

Table I. First Appearance of Prime Differences (Continued)

D	PA	PB	(LOG PB)/SQRT(D-1)
142	10343761	10343903	1,360
144	11981443	11981587	1,363
146	6034247	6034393	1,297
148*	2010733	2010881	1,197
150	13626257	13626407	1,346
152	8421251	8421403	1,298
154*	4652353	4652507	1,241
156	17983717	17983873	1,342
158	49269581	49269739	1,414
160	33803689	33803849	1,375
162	39175217	39175379	1,378
164	20285099	20285263	1,318
166	83751121	83751287	1,420
168	37305713	37305881	1,349
170	27915737	27915907	1,319
172	38394127	38394299	1,335
174	52721113	52721287	1,352
176	38089277	38089453	1,320
178	39389989	39390167	1,315
180*	17051707	17051887	1,245
182	36271601	36271783	1,294
184	79167733	79167917	1,344
186	147684137	147684323	1,383
188	134065829	134066017	1,368
190	142414669	142414859	1,366
192	123454691	123454883	1,348
194	166726367	166726561	1,363
196	70396393	70396589	1,294
198	46006769	46006967	1,257
200	378043979	378044179	1,400
202	107534587	107534789	1,304
204	112098817	112099021	1,301
206	232423823	232424029	1,345
208	192983851	192984059	1,326
210*	20831323	20831533	1,166

Table I. First Appearance of Prime Differences (Continued)

D	PA	PB	(LOG PB)/SQRT(D-1)
212	215949407	215949619	1,321
214	253878403	253878617	1,326
216	202551667	202551883	1,304
218	327966101	327966319	1,331
220 *	47326693	47326913	1,194
222 *	122164747	122164969	1,253
224	409866323	409866547	1,328
226	519653371	519653597	1,338
228	895858039	895858267	1,368
230	607010093	607010323	1,336
232	525436489	525436721	1,321
234 *	189695659	189695893	1,249
236	216668603	216668839	1,252
238	673919143	673919381	1,320
240	391995431	391995671	1,280
242	367876529	367876771	1,270
244	693103639	693103883	1,306
246	555142061	555142307	1,286
248 *	191912783	191913031	1,214
250 *	387096133	387096383	1,253
252	630045137	630045389	1,279
254	1202442089	1202442343	1,314
256	1872851947	1872852203	1,337
258	1316355323	1316355581	1,310
260	944192807	944193067	1,284
262	1649328997	1649329259	1,314
264	2357881993	2357882257	1,331
266	1438779821	1438780087	1,295
268	1579306789	1579307057	1,296
270	1391048047	1391048317	1,284
272	1851255191	1851255463	1,296
274	1282463269	1282463543	1,269
276	649580171	649580447	1,224
278	4260928601	4260928879	1,332
280	1855047163	1855047443	1,278

Table I. First Appearance of Prime Differences (Concluded)

D	PA	PB	(LOG PB)/SQRT(D-1)
282 *	436273009	436273291	1.187
284	1667186459	1667186743	1.262
286	2842739311	2842739597	1.289
288 *	1294268491	1294268779	1.238
290	1948819133	1948819423	1.258
292 *	1453168141	1453168433	1.237
294	5692630189	5692630483	1.312
296	5260030511	5260030807	1.303
298	8650524583	8650524881	1.328
300	4758958741	4758959041	1.289
302	6675573497	6675573799	1.304
304	2433630109	2433630413	1.242
306	3917587237	3917587543	1.265
308	5490459101	5490459409	1.280
310	4024713661	4024713971	1.258
312	6570018347	6570018659	1.282
314	8948418749	8948419063	1.295
318	4372999721	4373000039	1.247
320 *	2300942549	2300942869	1.207
322	7961074441	7961074763	1.272
324	10958687879	10958688203	1.286
326	5837935373	5837935699	1.247
330	6291356009	6291356339	1.244
332	5893180121	5893180453	1.237
336*	3842610773	3842611109	1.206
340	8605261447	8605261787	1.242
354*	4302407359	4302407713	1.181
382*	10726904659	10726905041	1.183

intervening numbers were composite. Of course, as soon as any eligible number being tested was found to be composite, it was rejected, and the next eligible number was selected for testing. Since there was no room in the program to store a table of pseudoprimes to the base 2, experiments with the converse of Fermat's Theorem to detect composite numbers were dropped when it was noted that the program spent the majority of its running time verifying the primality of the end points, rather than eliminating composite numbers between the end points.

With the availability of a larger computer memory in which to store a table of primes and their starting points with respect to a fixed field of bits, it becomes feasible to use a sieve technique for extending this search. However, with a very limited computer memory, the algorithm given above has the advantage of requiring only a table of previously found differences, and a starting point for each run, and thus could be used as a small background problem.

A program for the CDC 6600 was written to implement a sieve technique for generating and examining gaps in primes. This program occupied considerably more memory but ran significantly faster (partially due to an increase in computer speed) than the program described above. The sieve program allocated a block of computer memory in which consecutive bits represented the successive odd integers. A table of the first ten thousand primes was generated and stored by the program during initialization. Another table of starting points (i.e., index of the first bit in the field corresponding to a multiple of each prime in the stored table) for marking by each prime in the sieve field was also generated and saved. The program then cycled through successive bit fields marking bits corresponding to the odd composite numbers, then searched the field for gaps of interest. End effects at the boundaries of the sieve fields were noted so that gaps of interest would not be missed. Table I for $1.46 \times 10^9 < P < 1.096 \times 10^{10}$ presents the results obtained from this program.

In private correspondence Daniel Shanks suggested the possibility of extending Table I in [2] over the new differences found. Accordingly,

Table I shows $\log P_b / \sqrt{D-1}$, with each maximal gap D marked with an asterisk. Maximal gaps, according to Shanks, are those larger than any preceding gap in the sequence of primes. These data tend to support the conjectured relation in [2], namely that $\log P_b \sim \sqrt{D-1}$ for maximal gaps, and also, possibly, for all gaps at the point of their first appearance.

B. REFERENCES

1. Uspensky and Heaslet, Elementary Number Theory, McGraw-Hill, 1939, p. 90, paragraph 14.

2. Daniel Shanks, "On Maximal Gaps Between Successive Primes," Math. Comp., v. 18, 1964, pp. 646-650.

3. Selmer M. Johnson, "An Elementary Remark on Maximal Gaps Between Successive Primes," Math. Comp., v. 19, 1965, pp. 675-676.

4. C. L. Baker and F. J. Gruenberger, The First Six Million Prime Numbers, The RAND Corp., July 1957, published by the Microcard Foundation, West Salem, Wisconsin, 1959.

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III. CONSECUTIVE PRIMES IN ARITHMETIC PROGRESSION

A. DISCUSSION

A. Schinzel and W. Sierpinski [1] conjectured that there exist arbitrarily long arithmetic progressions formed of consecutive prime numbers. Sierpinski stated in [2] that a progression of five consecutive primes had not yet been found. A direct computer search showed that the first such progression has the common difference d = 30 and begins with the prime 9,843,019. The first progression of six consecutive primes begins with 121,174,811 and also has d = 30. Up to the limit 3×10^8 there are 25 other progressions of five consecutive primes, all with d = 30; there are no other progressions of six consecutive primes.

The reviewer points out that recently a much larger quintuplet, beginning with 10000024493, and again having d = 30, was recorded [3], but without reference to Sierpinski's remark. The smaller set that we found, and the single sextuplet, may still be worth recording.

B. REFERENCES

1. A. Schinzel and W. Sierpinski, "Sur certaines hypotheses concernant les nombres premiers," Acta Arith., v. 4, 1958, p. 191.

2. W. Sierpinski, A Selection of Problems in the Theory of Numbers, MacMillan, New York, 1964, p. 105.

3. M. F. Jones, M. Lal, and W. J. Blundon, "Statistics on Certain Large Primes," Math. Comp. (to appear).

IV. A COUNTEREXAMPLE TO EULER'S SUM OF POWERS CONJECTURE

A. DISCUSSION

A search was conducted on the CDC 6600 computer for non-trivial solutions in non-negative integers of the Diophantine equation

$$x_1^5 + x_2^5 + \cdots + x_n^5 = y^5 \quad , \quad n \leq 6. \tag{1}$$

In general, to decompose t as the sum of n fifth powers, assume s is the largest. Then for each s in the range

$$\left(\frac{t}{n}\right)^{1/5} \leq s \leq t^{1/5}$$

a decomposition is sought in which $t - s^5$ is the sum of $n - 1$ fifth powers each $\leq s^5$. Applying the algorithm repeatedly, a final decomposition is reached of the form

$$u = v^5 + w^5$$

in which $w \leq v$ and each v in the range $\left(\frac{u}{2}\right)^{1/5} \leq v \leq u^{1/5}$ is considered. Since $x^5 \equiv x \pmod{30}$ for each integer x, we require $w \equiv u - v \pmod{30}$. A precalculated table of fifth powers was employed and table lookup replaced the taking of fifth roots in determining limits on the x_i.

For $n = 6$, there are only ten primitive solutions of (1) in the range $y \leq 100$ and these are given in Table I. The least two of these were obtained by A. Martin (Reference 1). Among the new solutions was

$$19^5 + 43^5 + 46^5 + 47^5 + 67^5 = 72^5$$

which is the least solution of (1) with $n = 5$. The search was then specialized to $n = 5$, and the four solutions given in Table II are the only primitive solutions over the range $y \leq 250$. The third case ($y = 107$) is the least solution given by Sastry's identity (Reference 2)

$$(75\, v^5 - u^5)^5 + (u^5 + 25v^5)^5 + (u^5 - 25v^5)^5 + (10\, u^3v^2)^5 + (50\, uv^4)^5 = (u^5 + 75v^5)^5$$

for $u = 2$, $v = 1$.

The fourth case was the unexpected result

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

which is a counterexample to Euler's conjecture (Reference 3) that at least k positive k^{th} powers are required to sum to a k^{th} power, except for the trivial case of one k^{th} power: $y^k = y^k$. The search was again specialized to n = 4 over the range $y \leq 750$, but no further primitive solutions exist in that range.

Table II. All Primitive Solutions of $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_6^5 = y^5$, $y \leq 100$

x_1	x_2	x_3	x_4	x_5	x_6	y
4	5	6	7	9	11	12
5	10	11	16	19	29	30
15	16	17	22	24	28	32
13	18	23	31	36	66	67
7	20	29	31	34	66	67
0	19	43	46	47	67	72
22	35	48	58	61	64	78
0	21	23	37	79	84	94
4	13	19	20	67	96	99
6	17	60	64	73	89	99

Table III. All Primitive Solutions of $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = y^5$, $y \leq 250$

x_1	x_2	x_3	x_4	x_5	y
19	43	46	47	67	72
21	23	37	79	84	94
7	43	57	80	100	107
0	27	84	110	133	144

B. REFERENCES

1. A. Martin, Bull, Phil. Soc. Wash., 10, 1887, 107, in Smithsonian Miscel. Coll., 33, 1888.

2. S. Sastry, Journal London Math. Soc., 9 (1934), p. 242.

3. L. E. Dickson, History of the Theory of Numbers, v. 2, p. 682.

V. A SURVEY OF EQUAL SUMS OF LIKE POWERS

A. INTRODUCTION

The Diophantine equation

$$x_1^k + x_2^k + \cdots + x_m^k = y_1^k + y_2^k + \cdots y_n^k \qquad 1 \leq m \leq n \tag{1}$$

has been studied by numerous mathematicians for many years and by various methods [1], [2]. We recently conducted a series of computer searches using the CDC 6600 to identify the sets of parameters k, m, n for which solutions exist and to find the least solutions for certain sets. This section outlines the results of the computation, notes some previously published results, and concludes with a table showing, for various values of k and m, the least n for which a solution to (1) is known.

We restrict our attention to $k \leq 10$. We assume that the x_i and y_j are positive integers and $x_i \neq y_j$. We do not distinguish between solutions which differ only in that the x_i or y_j are rearranged. We will refer to (1) as (k. m. n) and say that a primitive solution to (k. m. n) is one in which no integer > 1 divides all the numbers $x_1, x_2, \cdots x_m, y_1, y_2, \cdots y_n$. Putting

$$z = \sum_1^m x_i^k = \sum_1^n y_j^k,$$

we order the primitive solutions according to the magnitude of z and denote the r^{th} primitive solution to (k. m. n) by $(k.\,m.\,n)_r$. Where we refer to the <u>range</u> covered in a search for solutions, we mean the upper limit on z. The notation $(x_1, x_2, \cdots, x_m)^k = (y_1, y_2, \cdots, y_n)^k$ means

$\sum_{1}^{m} x_i^k = \sum_{1}^{n} y_j^k$. Any parametric solution discussed does not include all solutions unless otherwise stated.

B. SQUARES AND CUBES

For $k = 2$ the general solution of the Pythagorean equation (2. 1. 2) is well known [2, pp. 165-170]. Many solutions in small integers and various parametric solutions have been given for (2. 1. n) with $n \geq 3$. The general solution of (2. 2. 2) is known [2, p. 252]. Solutions to (2. 2. n) with $n \geq 3$ and (2. m. n) with $m \geq 3$ are numerous.

The impossibility of solving (k. 1. 2 with $k \geq 3$ is Fermat's last theorem, which has been established for $k \leq 25000$ [3]. The general solution of (3. 1. 3) in rationals is attributed to Euler and Vieta [2, p. 550-554] and also produces all solutions to (3. 2. 2) if the arguments are properly chosen. There are many solutions in small integers and various parametric solutions to (3. 1. n) with $n \geq 4$ and to (3. m. n) with $m \geq 2$ [2, pp. 563-565].

C. FOURTH POWERS

(4. 1. n) -- For $n = 3$, no solution is known. M. Ward [4] developed congruential constraints which, together with some hand computing, allowed him to show that $x^4 = y_1^4 + y_2^4 + y_3^4$ has no solution if $x \leq 10,000$. The authors extended the search on the computer using a similar method and verified that there is no solution for $x \leq 220,000$. Ward showed that if $x^4 = y_1^4 + y_2^4 + y_3^4$ is a primitive solution, it may be assumed that $x, y_1 \equiv 1 \pmod 2$, $y_2, y_3 \equiv 0 \pmod 8$ and either $x - y_1$ or $x + y_1$ is $\equiv 0 \pmod{1024}$. Also $x \not\equiv 0 \pmod 5$, or else all y_i would be $\equiv 0 \pmod 5$, since $u^4 \equiv 0$ or 1 according as $u \equiv 0$ or $u \not\equiv 0 \pmod 5$. The computer program generated

all numbers $M = (x^4 - y_1^4)/2048$ with $0 < y_1 < x$, x prime to 10 and $y_1 \equiv \pm x$ (mod 1024). Tests were applied to $M = (y_2/8)^4 + (y_3/8)^4$ to reject cases in which a solution would not be primitive or M could not be the sum of two biquadrates. If M passed all the tests, its decomposition was attempted by trial using addition of entries in a stored table of biquadrates (27500 entries for $x \leq 220,000 = 8 \cdot 27500$). The tests were

(1) M must be $\equiv 0$, 1, or 2 (mod 16) and (mod 5)

(2) M must not be $\equiv 7$, 8, or 11 (mod 13) and must not be $\equiv 4$, 5, 6, 9, 13, 22 or 28 (mod 29)

(3) x and y_1 must not both be divisible by an odd prime $p \equiv 3$, 5 or 7 (mod 8) for if so, p^4 divides M, p divides y_2 and y_3, and the solution is not primitive

(4) M must not have a factor p where p is an odd prime not $\equiv 1$ (mod 8) unless p^4 also divides M. In this case p divides y^2 and y^3, and in the decomposition by trial M can be replaced by M/p^4 (here tests were made only for $p < 100$).

Of approximately 19,200,000 initial values of M, only 22,400 required the trial decomposition.

For n = 4, R. Norrie [5] found the smallest solution $(353)^4 = (30, 120, 272, 315)^4$. J. O. Patterson [28] found $(4.1.4)_2$ and J. Leech [6] found the next six primitive solutions on the EDSAC 2 computer. S. Brudno [7] gave another primitive solution, the 14th in our Table IV. The authors exhaustively searched the range 8002^4 using Leech's method of finding the 23 primitives listed in Table IV. No parametric solution has been found for (4.1.4), although the general solution is known for (3.1.3) and a parametric solution (discussed later) is known for (5.1.5).

Table IV

Primitive Solutions of (4. 1. 4) for $z \leq (8002)^4$

$$z = x_1^4 = \sum_1^4 y_j^4$$

i	x_1	y_1	y_2	y_3	y_4	Ref.
1	353	30	120	272	315	[5]
2	651	240	340	430	599	[28]
3	2487	435	710	1384	2420	[6]
4	2501	1130	1190	1432	2365	[6]
5	2829	850	1010	1546	2745	[6]
6	3723	2270	2345	2460	3152	[6]
7	3973	350	1652	3230	3395	[6]
8	4267	205	1060	2650	4094	[6]
9	4333	1394	1750	3545	3670	
10	4449	699	700	2840	4250	
11	4949	380	1660	1880	4907	
12	5281	1000	1120	3233	5080	
13	5463	410	1412	3910	5055	
14	5491	955	1770	2634	5400	[7]
15	5543	30	1680	3043	5400	
16	5729	1354	1810	4355	5150	
17	6167	542	2770	4280	5695	
18	6609	50	885	5000	5984	
19	6801	1490	3468	4790	6185	
20	7101	1390	2850	5365	6368	
21	7209	160	1345	2790	7166	
22	7339	800	3052	5440	6635	
23	7703	2230	3196	5620	6995	

For $n \geq 5$ there exist many solutions in small integers. $(4.1.5)_1$ is $(5)^4 = (2,2,3,4,4)^4$. Several parametric solutions to (4.1.5) are known due to E. Fauquembergue [8], C. Haldeman [9], and A. Martin [10].

(4.2.n) -- For $n = 2$ the least solution is $(59,158)^4 = (133,134)^4$. Euler [11] gave a two-parameter solution and A. Gerardin [12] gave an equivalent but simpler form of this solution. Several of the smaller primitive solutions were found by Euler, A. Werebrusow, and Leech [13], and a recent computer search by Lander and Parkin [14] extended the list of known primitives to 31. More recently we have increased this to a total of 46 primitives by a complete search of the range 5.3×10^{16} and the 15 new primitives are listed in Table V. The general solution is not known.

For $n \geq 3$ there are many small solutions. $(4.2.3)_1$ is $(7,7)^4 = (3,5,8)^4$. Several parametric solutions are known for (4.2.3) due to Gerardin [15] and F. Ferrari [16].

(4.m.n) -- For $m \geq 3$, solutions in small integers are numerous. Parametric solutions to (4.3.3) were given by Gérardin [17] and Werebrusow [18]. $(4.3.3)_1$ is $(2,4,7)^4 = (3\ 6,6)^4$.

D. <u>FIFTH POWERS</u>

(5.1.n) -- For $n = 3$, no solution is known. Lander and Parkin [19], [20] found $(5.1.4)_1$ to be $(144)^5 = (27,84,110,133)^5$. This disproved Euler's conjecture [2, p. 648] that (k.1.n) has no solution if $1 < n < k$. No further primitive solutions to (5.1.4) exist in the range up to 765^5.

For $n = 5$, S. Sastry and S. Chowla [21] obtained a two-parameter solution yielding $(107)^5 = (7,43,57,80,100)^5$ as its minimal primitive; this solution

Table V

Primitive Solutions of (4. 2. 2) for $7.5 \times 10^{15} \le z \le 5.3 \times 10^{16}$

$$z = x_1^4 + x_2^4 = y_1^4 + y_2^4$$

i	x_1	x_2	y_1	y_2	z
*32	6262	8961	7234	8511	7 98564 45223 00177
33	5452	9733	7528	9029	9 85755 13638 85937
34	3401	10142	7054	9527	10 71400 42234 80497
35	5277	10409	8103	9517	12 51457 36160 92402
36	3779	10652	8332	9533	13 07827 22453 98097
37	3644	11515	5960	11333	17 75781 85225 58321
38	1525	12234	3550	12213	22 40674 37332 52161
**39	2903	12231	10203	10381	22 45039 16406 17602
40	1149	12653	7809	12167	25 63324 34950 11682
41	5121	13472	9153	12772	33 62808 84147 85537
42	5526	13751	11022	12169	36 68751 70593 08977
43	6470	14421	8171	14190	45 00187 64129 98081
44	6496	14643	11379	13268	47 75551 49900 03857
45	261	14861	8427	14461	48 77442 72266 31682
46	581	15109	8461	14723	52 11273 11403 26882

* For solutions to (4. 2. 2) for i = 1 to 31 see Lander and Parkin [14].

** This solution was found by Euler [2, p. 644].

is $(5.1.5)_3$. Lander and Parkin [20] found $(5.1.5)_1$ and $(5.1.5)_2$ to be $(72)^5 = (19, 43, 46, 47, 67)^5$ and $(94)^5 = (21, 23, 37, 79, 84)^5$. More recently we searched the range up to 599^5 and found the twelve primitive solutions given in Table VI.

For $n \geq 6$ there are solutions in moderately small integers. $(5.1.6)_1$ is $(12)^5 = (4, 5, 6, 7, 9, 11)^5$ found by A. Martin [22]. The first eight primitive solutions to (5.1.6) are given in [20]. $(5.1.7)_1$ is $(23)^5 = (1, 7, 8, 14, 15, 18, 20)^5$.

(5.2.n) -- No solution is known for $n \leq 3$. An exhaustive search by the authors verified that there is no solution to (5.2.2) in the range up to 2.8×10^{14} or to (5.2.3) in the range up to 8×10^{12}. Sastry's parametric solution for (5.1.5) mentioned above gives for certain values of its arguments solutions to (5.2.4), the smallest being $(12, 38)^5 = (5, 13, 25, 37)^5$, which is $(5.2.4)_2$. K. Subba Rao [23] found $(3, 29)^5 = (4, 10, 20, 28)^5$, which is $(5.2.4)_1$. Table VII lists the ten primitives which exist in the range up to 2×10^{10}.

For $n \geq 5$, there are solutions in moderately small integers. $(5.2.5)_1$ is $(1, 22)^5 = (4, 5, 7, 16, 21)^5$ due to Subba Rao [23]. We give the first six primitives for (5.2.5) in Table VIII.

(5.3.n) -- The first solution known for $n = 3$ was $(49, 75, 107)^5 = (39, 92, 100)^5$ due to A. Moessner [29]; this is $(5.3.3)_5$. H. P. F. Swinnerton-Dyer gave two separate, two-parameter solutions [30]. We give the 45 primitives in the range up to 8×10^{12} in Table IX. For $n \geq 4$, solutions in small integers are plentiful. $(5.3.4)_1$ is $(3, 22, 25)^5 = (1, 8, 14, 27)^5$ due to Subba Rao [23]. A two-parameter solution to (5.3.4) was given by G. Xeroudakes and A. Moessner [24].

Table VI

Primitive Solutions of (5. 1. 5) for $z \leq 599^5$

$$z = x_1^5 = \sum_1^5 y_j^5$$

i	x_1	y_1	y_2	y_3	y_4	y_5	Ref.
1	72	19	43	46	47	67	[20]
2	94	21	23	37	79	84	[20]
3	107	7	43	57	80	100	[21]
4	365	78	120	191	259	347	
5	415	79	202	258	261	395	
6	427	4	26	139	296	412	
7	435	31	105	139	314	416	
8	480	54	91	101	404	430	
9	503	19	201	347	388	448	
10	530	159	172	200	356	513	
11	553	218	276	385	409	495	
12	575	2	298	351	474	500	

Table VII

Primitive Solutions of (5. 2. 4) for $z \leq 2 \times 10^{10}$

$$z = \sum_{1}^{2} x_j^5 = \sum_{1}^{4} y_j^5$$

i	x_1	x_2	y_1	y_2	y_3	y_4	z	Ref.
1	3	29	4	10	20	28	205 11392	[23]
2	12	38	5	13	25	37	794 84000	[21]
3	28	52	26	29	35	50	3974 14400	
4	61	64	5	25	62	63	19183 38125	
5	16	85	6	50	53	82	44381 01701	
6	31	96	56	63	72	86	81823 56127	
7	14	99	44	58	67	94	95104 38323	
8	63	97	11	13	37	99	95797 76800	
9	25	106	48	57	76	100	1 33920 21401	
10	54	111	58	76	79	102	1 73097 46575	

Table VIII

Primitive Solutions of (5.2.5) for $z \leq 2.8 \times 10^8$

$$z = \sum_1^2 x_j^5 = \sum_1^5 y_j^5$$

i	x_1	x_2	y_1	y_2	y_3	y_4	y_5	z
*1	1	22	4	5	7	16	21	51 53633
2	23	29	9	11	14	18	30	269 47492
3	16	38	10	14	26	31	33	802 83744
4	24	42	4	22	29	35	36	1386 53856
5	30	44	8	15	17	19	45	1892 16224
6	36	42	5	6	26	27	44	1911 57408

* The first solution is due to Subba Rao [23].

Table IX

Primitive Solutions of (5. 3. 3) for $z \leq 8 \times 10^{12}$

$$z = \sum_{1}^{3} x_j^5 = \sum_{1}^{3} y_j^5$$

i	x_1	x_2	x_3	y_1	y_2	y_3	z
1	24	28	67	3	54	62	13752 98099
2	18	44	66	13	51	64	14191 38368
3	21	43	74	8	62	68	23700 99168
4	56	67	83	53	72	81	58398 97526
5*	49	75	107	39	92	100	1 66810 39431
6	26	85	118	53	90	116	2 73265 12069
7	38	47	123	1	89	118	2 84616 37018
8	73	96	119	68	106	114	3 40903 35168
9	39	56	136	3	97	131	4 71668 30151
10	13	35	142	17	95	138	5 77882 32400
11	28	32	155	91	94	150	8 95168 61675
12	65	94	152	42	129	140	8 96361 42881
13	63	67	169	9	131	159	14 02010 53499
14	68	137	170	36	140	169	19 17013 58025
15	43	109	181	13	159	161	20 97974 92893
16	74	113	182	61	129	179	22 03336 44849
17	39	142	186	28	167	172	28 04458 41607
18	44	55	201	18	152	190	32 87486 01600
19	58	101	204	113	145	195	36 44723 14293
20	18	31	215	10	183	191	45 94319 03094
21	19	168	216	11	183	209	60 40152 82243
22	5	145	224	153	157	214	62 80466 82374
23	27	106	229	12	122	228	64 31599 96832
24	151	166	233	126	208	216	89 12718 82720
25	59	139	248	23	184	239	99 07237 88960

* This solution was found by A. Moessner [29];

Table IX (Concluded)

Primitive Solutions of (5.3.3) for $z \leq 8 \times 10^{12}$

$$z = \Sigma x_j^5 = \Sigma y_j^5$$

i	x_1	x_2	x_3	y_1	y_2	y_3	z
26	157	193	234	147	218	219	106 47575 48174
27	2	97	258	35	125	257	115 17249 93057
28	3	121	264	163	185	250	130 83259 82668
29	97	181	274	67	227	258	174 72267 67782
30	99	105	286	30	179	281	193 57802 02300
31	132	154	283	80	219	270	194 19238 97099
32	106	137	288	201	219	261	204 29996 35401
33	40	168	289	3	215	279	214 99241 22017
34	136	158	294	71	249	268	234 15192 15168
35	193	229	282	179	259	266	268 09353 50774
36	107	229	293	93	259	277	280 32137 94149
37	31	173	307	7	201	303	288 20348 39551
38	102	118	310	49	270	271	289 68334 65600
39	116	124	310	21	235	294	291 32347 67200
40	30	39	331	65	224	321	397 33103 34850
41	119	232	328	89	289	301	449 23488 61399
42	108	181	348	53	246	338	531 27877 53637
43	114	211	364	52	298	339	682 75705 13699
44	172	206	364	102	303	337	691 15935 15232
45	123	137	373	13	259	361	729 65305 14393

(5. m. n) -- If $m \geq 4$, there are many solutions in small integers. $(5.4.4)_1$ is $(5, 6, 6, 8)^5 = (4, 7, 7, 7)^5$ due to Subba Rao [23]. Several parametric solutions to (5. 4. 4) were found by Xeroudakes and Moessner [24]. The first triple coincidence of four fifth powers is $1479604544 = (3, 48, 52, 61)^5 = (13, 36, 51, 64)^5 = (18, 36, 44, 66)^5$.

In the subsequent discussion we adopt a notation borrowed from the field of partitions, writing x^r to signify the term x repeated r times in the expression in which it appears. Table X uses this notation, giving $(k.m.n)_1$ where known and referencing solutions in other tables. Table X also shows for certain (k. m. n) the range which has been searched on the computer exhaustively.

For the remainder of the equations (k. m. n) which are discussed, we note in the text only the limits searched, interesting features, and methods employed, specific solutions are given in Table X.

E. SIXTH POWERS

(6. 1. n) -- No solution is known for $n \leq 6$. We consider the cases of $n = 6$, 7, and 8 in descending order. To solve (6. 1. 8), $x^6 = \sum_1^8 y_i^6$, note that $u^6 \equiv 0$ or $1 \pmod 9$ according to whether $u \equiv 0$ or $u \not\equiv 0 \pmod 3$. Then if $x \equiv 0 \pmod 3$, all $y_i \equiv 0 \pmod 3$ and the solution is not primitive. Therefore, take x and exactly one of the y_i (say y_1) prime to 3. Then $(x^6 - y_1^6)/3^6 = \sum_2^8 (y_i/3)^6$ is an integer (which is true if and only if $y_1 \equiv \pm x \pmod{243}$) to be decomposed by trial as the sum of 7 sixth powers. In Table XI, we

Table X

$(k.m.n)_1$ and Summary of Results

(k. m. n)	Range Searched	Solutions Known*
4.1.3	2.34×10^{21}	None known
4.1.4	4.1×10^{15}	$(353)^4 = (30, 120, 272, 315)^4$
		See Table IV, 23 solutions
4.1.5		$(5)^4 = (2^2, 3, 4^2)^4$
4.2.2	5.3×10^{16}	$(59, 158)^4 = (133, 134)^4$
		See Table I in [18], and Table V, 46 solutions
4.2.3		$(7^2)^4 = (3, 5, 8)^4$
4.3.3		$(2, 4, 7)^4 = (3, 6^2)^4$
5.1.3	2.6×10^{14}	None known
5.1.4	2.6×10^{14}	$(144)^5 = (27, 84, 110, 133)^5$
5.1.5	7.7×10^{13}	$(72)^5 = (19, 43, 46, 47, 67)^5$
		See Table VI, 12 solutions
5.1.6		$(12)^5 = (4, 5, 6, 7, 9, 11)^5$
5.1.7		$(23)^5 = (1, 7, 8, 14, 15, 18, 20)^5$
5.2.2	2.8×10^{14}	None known
5.2.3	8×10^{12}	None known
5.2.4	2×10^{10}	$(3, 29)^5 = (4, 10, 20, 28)^5$
		See Table VII, 10 solutions
5.2.5	2×10^{8}	$(1, 22)^5 = (4, 5, 7, 16, 21)^5$
		See Table VIII, 6 solutions
5.3.3	8×10^{12}	$(24, 28, 67)^5 = (3, 54, 62)^5$
		See Table IX, 45 solutions

* All solutions shown are $(k.m.n)_1$ unless otherwise marked.

Table X (Continued)

$(k.\ m.\ n)_1$ and Summary of Results

(k. m. n)	Range Searched	Solutions Known
5. 3. 4		$(3, 22, 25)^5 = (1, 8, 14, 27)^5$
5. 4. 4		$(5, 6^2, 8)^5 = (4, 7^3)^5$
6. 1. n	3.16×10^{27}	None known for $n \le 6$
6. 1. 7	1.3×10^{19}	$(1141)^6 = (74, 234, 402, 474, 702, 894, 1077)^6$
6. 1. 8	5.8×10^{16}	$(251)^6 = (8, 12, 30, 78, 102, 138, 165, 246)^6$ See Table XI, 14 solutions
6. 1. 9		$(54)^6 = (1, 17, 19, 22, 31, 37^2, 41, 49)^6$
6. 1. 10		$(39)^6 = (2, 4, 7, 14, 16, 26^2, 30, 32^2)^6$
6. 1. 11		$(18)^6 = (2, 5^3, 7^2, 9^2, 10, 14, 17)^6$
6. 2. n	4×10^{12}	None known for $n \le 6$
6. 2. 7		$(56, 91)^6 = (18, 22, 36, 58, 69, 78^2)^6$
6. 2. 8		$(35, 37)^6 = (8, 10, 12, 15, 24, 30, 33, 36)^6$
6. 2. 9		$(6, 21)^6 = (1, 5^2, 7, 13^3, 17, 19)^6$
6. 2. 10		$(12^2)^6 = (1^3, 4^2, 7, 9, 11^3)^6$
6. 3. 3	2.5×10^{14}	$(3, 19, 22)^6 = (10, 15, 23)^6$ See Table XII, 10 solutions
6. 3. 4	2.9×10^{12}	$(41, 58, 73)^6 = (15, 32, 65, 70)^6$ See Table XIII, 5 solutions
6. 4. 4		$(2^2, 9^2)^6 = (3, 5, 6, 10)^6$
7. 1. n	1.95×10^{14}	None known for $n \le 7$
7. 1. 8		$(102)^7 = (12, 35, 53, 58.64, 83, 85, 90)^7$
7. 1. 9		$(62)^7 = (6, 14, 20, 22, 27, 33.41, 50, 59)^7$
7. 2. 8		$(10, 33)^7 = (5, 6, 7, 15^2, 20, 28, 31)^7$
7. 3. 7		$(26, 30^2)^7 = (7^2, 12, 16, 27, 28, 31)^7$
7. 4. 5		$(12, 16, 43, 50)^7 = (3, 11, 26, 29, 52)^7$

Table X (Continued)

(k. m. n)$_1$ and Summary of Results

(k. m. n)	Range Searched	Solutions Known
7. 5. 5		$(8^2, 13, 16, 19)^7 = (2, 12, 15, 17, 18)^7$
		See Table XIV, 17 solutions
7. 6. 6		$(2, 3, 6^2, 10, 13)^7 = (1^2, 7^2, 12^2)^7$
8. 1. 11		$(125)^8 = (14, 18, 44^2, 66, 70, 92, 93, 96, 106, 112)^8$
8. 1. 12		$(65)^8 = (8^2, 10, 24^3, 26, 30, 34, 44, 52, 63)^8$
8. 2. 9		$(11, 27)^8 = (2, 7, 8, 16, 17, 20^2, 24^2)^8$
8. 3. 8		$(8, 17, 50)^8 = (6, 12, 16^2, 38^2, 40, 47)^8$
8. 4. 7		$(6, 11, 20, 35)^8 = (7, 9, 16, 22^2, 28, 34)^8$
8. 5. 5		$(1, 10, 11, 20, 43)^8 = (5, 28, 32, 35, 41)^8$
8. 6. 6		$(3, 6, 8, 10, 15, 23)^8 = (5, 9^2, 12, 20, 22)^8$
8. 7. 7		$(1, 3, 5, 6^2, 8, 13)^8 = (4, 7, 9^2, 10, 11, 12)^8$
8. 8. 8		$(1, 3, 7^3, 10^2, 12)^8 = (4, 5^2, 6^2, 11^3)^8$

Table X (Concluded)

$(k.\ m.\ n)_1$ and Summary of Results

(k. m. n)	Range Searched	Solutions Known
9. 1. 15		$(26)^9 = (2^2, 4, 6^2, 7, 9^2, 10, 15, 18, 21^2, 23^2)^9$
9. 2. 12		$(15, 21)^9 = (2^4, 3^2, 4, 7, 16, 17, 19^2)^9$
9. 3. 11		$(13, 16, 30)^9 = (2, 3, 6, 7, 9^2, 19^2, 21, 25, 29)^9$
9. 4. 10		$(5, 12, 16, 21)^9 = (2, 6^2, 9, 10, 11, 14, 18, 19^2)^9$
9. 5. 11		$(7, 8, 14, 20, 22)^9 = (3, 5^2, 9^2, 12, 15^2, 16, 21^2)^9$
9. 6. 6		$(1, 13^2, 14, 18, 23)^9 = (5, 9, 10, 15, 21, 22)^9$
10. 1. 23		$(15)^{10} = (1^5, 2, 3, 6, 7^6, 9^4, 10, 12^2, 13, 14)^{10}$
10. 2. 19		$(9, 17)^{10} = (2^5, 5, 6, 10, 11^6, 12^2, 15^3)^{10}$
10. 3. 24		$(11, 15^2)^{10} = (1, 2, 3, 4^{10}, 7, 8^7, 10, 12, 16)^{10}$
10. 4. 23		$(11^3, 16)^{10} = (1^5, 2^2, 3^2, 4, 6^4, 7^3, 8, 10^2, 14^2, 15)^{10}$
10. 5. 16		$(3^2, 8, 14, 16)^{10} = (1^4, 2, 4^2, 6, 12^2, 13^5, 15)^{10}$
10. 6. 27		$(2^2, 8, 11\ 12^2)^{10} = (1, 3^4, 4^2, 5^2, 6^7, 7^9, 10, 13)^{10}$
10. 7. 7*		$(1, 28, 31, 32, 55, 61, 68)^{10} = (17, 20, 23, 44, 49, 64, 67)^{10}$

*Moessner [29]; not known to be $(10.\ 7.\ 7)_1$.

Table XI

Primitive Solutions of (6. 1. 8) for $z \leq 7 \times 10^{16}$

$$z = x_1^6 = \sum_1^8 y_i^6$$

i	x_1	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
1	251	8	12	30	78	102	138	165	246
2	431	48	111	156	186	188	228	240	426
3	440	93	93	195	197	303	303	303	411
4	440	219	255	261	267	289	351	351	351
5	455	12	66	138	174	212	288	306	441
6	493	12	48	222	236	333	384	390	426
7	499	66	78	144	228	256	288	435	444
8	502	16	24	60	156	204	276	330	492
9	547	61	96	156	228	276	318	354	534
10	559	170	177	276	312	312	408	450	498
11	581	60	102	126	261	270	338	354	570
12	583	57	146	150	360	390	402	444	528
13	607	33	72	122	192	204	390	534	534
14	623	12	90	114	114	273	306	492	592

give the 14 smallest primitives found by this method; $(6.1.8)_1$ is $(251)^6 = (8,12,30,78,102,138,165,246)^6$.

For (6.1.7), $x^6 = \sum_1^7 y_i^6$, note that $u^6 \equiv 0$ or 1 (mod 8) according as u is even or odd. Then for a primitive solution, x and exactly one of the y_i are odd. The argument for (6.1.8) modulo 9 applies and x is prime to 6, y_1 (say) is prime to 3, and either y_1 is odd or another y (say y_2) is odd. In the first case $y_1 \equiv \pm x$ (mod 243) and (mod 32), and $(x^6 - y_1^6)/6^6 = \sum_2^7 (y_i/6)^6$ is an integer to be decomposed by trial as the sum of 6 sixth powers. In the second case $y_1 \equiv \pm x$ (mod 243), $y_2 \equiv \pm x$ (mod 32) and $(x^6 - y_1^6 - y_2^6)/6^6 = \sum_3^7 (y_i/6)^6$ must be an integer (certain combinations x, y_1, y_2 satisfying the congruences are rejected), which is decomposed by trial as the sum of 5 sixth powers. The only solution for $x \leq 1536$ is $(6.1.7)_1$, $(1141)^6 = (74,234,402,474,702,894,1077)^6$, which is obtained in the second case.

For (6.1.6), $x^6 = \sum_1^6 y_i^6$ note that $u^6 \equiv 0$ or 1 (mod 7) according as $u \equiv 0$ or $u \not\equiv 0$ (mod 7). Then for a primitive solution, x and exactly one of the y_i (say y_1) are prime to 7. This implies $y_1 \equiv \pm x$, $\pm qx$ or $\pm q^2x$ where $q = 34968$ is a primitive sixth root of unity (mod $7^6 = 117649$). Now the foregoing arguments modulo 8 and modulo 9 apply, and there are five cases.

(1) If $y_1 \equiv \pm 1$ (mod 6), then $y_1 \equiv \pm x$ (mod 243) and (mod 32) and $(x^6 - y_1^6)/42^6 = \sum_2^6 (y_i/42)^6$ is an integer to be decomposed by trial as the sum of 5 sixth powers.

(2) If $y_1 \equiv \pm 2$ (mod 6), then $y_1 \equiv \pm x$ (mod 243) and another of the y_1 (say y_2) is odd. Then $y_2 \equiv 0$ (mod $3 \cdot 7$), $y_2 \equiv \pm x$ (mod 32), and $(x^6 - y_1^6 - y_2^6)/42^6 = \sum_3^6 (y_i/42)^6$ is the sum of 4 integral sixth powers.

(3) If $y_1 \equiv 3 \pmod 6$, then $y_1 \equiv \pm x \pmod{32}$ and another of the y_i (say y_2) is prime to 3, $y_2 \equiv 0 \pmod{2\cdot 7}$, and $y_2 \equiv \pm x \pmod{243}$. In case (2), $(x^6 - y_1^6 - y_2^6)/42^6$ is an integer and is the sum of 4 sixth powers.

(4) If $y_1 \equiv 0 \pmod 6$, another of the y_i (say y_2) is prime to 3, $y_2 \equiv 0 \pmod 7$, and $y_2 \equiv \pm x \pmod{243}$. If y_2 is odd, then $y_2 \equiv \pm x \pmod{32}$, and as in cases (2) and (3) $(x^6 - y_1^6 - y_2^6)/42^6$ is the sum of 4 sixth powers. If y_2 is even, we have case (5).

(5) Another of the y_i (say y_3) is odd, $y_3 \equiv 0 \pmod{3\cdot 7}$, $y_3 \equiv \pm x \pmod{32}$, and $(x^6 - y_1^6 - y_2^6 - y_3^6)/42^6 = \sum_4^6 (y_i/42)^6$ is an integer to be decomposed as the sum of 3 sixth powers.

The search for a solution to (6. 1. 6) was carried exhaustively by this method through the range $x \leq 38314$, and there is no solution in this range.

A. Martin [25] gave a solution to (6. 1. 16); Moessner [26] gave solutions to (6. 1. n) for n = 16, 18, 20 and 23. For $n \geq 11$, it is not difficult to find solutions in small integers.

(6. 3. n) -- Subba Rao [23] found the solution $(3, 19, 22)^6 = (10, 15, 23)^6$, which is $(6. 3. 3)_1$. In Table XII we give the remaining 9 primitive solutions which exist in the range up to 2.5×10^{14}. It is interesting to note that each of the solutions except the sixth is also a solution to (2. 3. 3). Table XIII gives the five primitive solutions to (6. 3. 4) which exist in the range up to 2.9×10^{12}.

(6. m. n) -- If m is ≥ 4, solutions in small integers can be found readily. Subba Rao [23] gave $(6. 4. 4)_1$ (see Table X. The first triple coincidence of four sixth powers is $1885800643779 = (1, 34, 49, 111)^6 = (7, 43, 69, 110)^6 = (18, 25, 77, 109)^6$.

Table XII

Primitive Solutions of (6. 3. 3) for $z \le 2.5 \times 10^{14}$

$$z = \sum_{1}^{3} x_j^6 = \sum_{1}^{3} y_j^6$$

i	x_1	x_2	x_3	y_1	y_2	y_3	z
1*	3	19	22	10	15	23	1604 26514
2	36	37	67	15	52	65	9 52008 90914
3	33	47	74	23	54	73	17 62771 73474
4	32	43	81	3	55	80	28 98246 41354
5	37	50	81	11	65	78	30 06202 62890
6	25	62	138	82	92	135	696 38068 13393
7	51	113	136	40	125	129	842 70669 28346
8	71	92	147	1	132	133	1082 47536 54794
9	111	121	230	26	169	225	15304 47319 28882
10	75	142	245	14	163	243	22464 65092 02194

* The first solution is due to K. Subba Rao [23].

Table XIII

Primitive Solutions of (6. 3. 4) for $z \le 2.9 \times 10^{12}$

$$z = \sum_{1}^{3} x_j^6 = \sum_{1}^{4} y_j^6$$

i	x_1	x_2	x_3	y_1	y_2	y_3	y_4	z
1	41	58	73	15	32	65	70	19 41530 23074
2	61	62	85	52	56	69	83	48 54701 25570
3	61	74	85	26	56	71	87	59 28763 80162
4	11	88	90	21	74	78	92	99 58468 58345
5	26	83	95	23	24	28	101	106 23411 79770

F. SEVENTH POWERS

$(7.2.10)_2$ is $(2, 27)^7 = (4, 8, 13, 14^2, 16, 18, 22, 23^2)^7 = (7^2, 9, 13, 14, 18, 20, 22^2, 23)^7$ which is a double primitive and reduces to the solution $(7.5.5)_2$.

(7.5.n) -- Table XIV lists the 17 primitive solutions to (7.5.5) which exist in the range up to 4.0×10^{12}.

G. EIGHTH POWERS

(8.1.n) -- We found a parametric solution to (8.1.17), $(2^{8k+4} + 1)^8 = (2^{8k+4} - 1)^8 + (2^{7k+4})^8 + (2^{k+1})^8 + 7\,[(2^{5k+3})^8 + (2^{3k+2})^8]$ which for $k = 0$ yields $(8.1.17)_1$. This was the solution used by Sastry [21] in developing a parametric solution to (8.8.8). The computer program used in searching for solutions to (8.1.n) was based on the congruences $x^8 \equiv 0$ or 1 (mod 32) according to whether $x \equiv 0$ or 1 (mod 2), so that primitive solutions to $x^8 = \sum_1^n y_j^8$ with $n < 32$ must have x and (say) y_1 both odd. Then $x^8 - y_1^8$ is divisible by 2^8, which implies $x \equiv \pm y_1$ (mod 32), and $(x^8 - y_1^8)/256$ is decomposed as the sum of n - 1 eighth powers by trial.

Solutions to (8.5.5) and (8.9.9) were found by A. Letac [27].

H. NINTH AND TENTH POWERS

Computations performed by the authors for (9.m.n) and (10.m.n) are the basis for the data shown in the last two columns of Table XV, except for a solution to (10.7.7) given by A. Moessner [29]. Due to computer word length limitations the calculations were not extended to large values of the arguments.

Table XIV

Primitive Solutions of (7, 5, 5) for $z \le 4.0 \times 10^{12}$

$$z = \sum_{1}^{5} x_j^7 = \sum_{1}^{5} y_j^7$$

i	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	z
1	8	8	13	16	19	2	12	15	17	18	12292 50016
2	4	8	14	16	23	7	7	9	20	22	37807 87943
3	11	12	18	21	26	9	10	22	23	24	1 05004 37728
4	6	12	20	22	27	10	13	13	25	26	1 42708 22835
5	3	13	17	24	38	14	26	32	32	33	11 94751 43393
6	4	5	30	36	44	2	8	27	39	43	41 95120 68269
7	16	33	33	33	44	18	26	34	38	43	44 74015 74051
8	3	4	21	39	45	14	23	33	41	43	51 27015 66916
9	16	17	26	33	49	10	12	30	43	46	72 95521 00131
10	15	18	18	43	48	8	11	32	44	47	86 02822 52818
11	19	24	43	46	51	9	36	40	48	50	161 05272 89337
12	13	16	35	35	56	9	19	28	44	55	185 61046 27259
13	9	11	43	45	55	3	19	37	51	53	216 79475 68747
14	9	15	19	34	59	5	10	16	48	57	254 22443 49046
15	23	27	40	49	56	7	39	45	51	53	258 30231 01035
16	8	13	41	45	59	2	10	47	52	55	305 71400 57494
17	1	38	39	39	60	8	25	34	53	57	318 82375 95951

Table XV

Least n for Which a Solution to (k. m. n) Is Known

m \ k	2	3	4	5	6	7	8	9	10
1	2	3	4	4	7	8	11	15	23
2	2	2	2	4	7	8	9	12	19
3				3	3	7	8	11	24
4						5	7	10	23
5						5	5	11	16
6								6	27
7									7

I. CONCLUDING REMARKS

Let N (k, m) be the smallest n for which (k. m. n) is solvable. In Table XV we show the upper bound to N based on the results just presented. Each column is terminated when a solution to (k. m. m) has been found. It appears likely that whenever (k. m. m) is solvable, so is (k. r. r) for any $r > m$. Some questions are:

a. Is $N(k, m+1) \le N(k, m) \le N(k+1, m)$ always true?

b. Is (k. m. n) always solvable when $m + n > k$?

c. Is it true that (k. m. n) is never solvable when $m + n < k$?

d. For which k, m, n, such that $m + n = k$, is (k. m. n) solvable?

The results presented in this section tend to support an affirmative answer to (c). Question (d) appears to be especially difficult. The only solvable cases with $m + n = k$ known at present are (4. 2. 2), (5. 1. 4), and (6. 3. 3).

In this section we have made a computational attack on the problem of finding a sum of n k^{th} powers which is also the sum of a smaller number of k^{th} powers. In many of the cases considered, especially for the larger values of k, we have undoubtedly not obtained the best possible results, but the amount of computing needed to do this would seem to be overwhelming.

We believe that the main result of this section is the presentation of results on a family of Diophantine equations which have largely been considered separately in the past. We hope that this presentation offers greater insight into the nature of the function N (k, m) and that the future efforts will be directed toward reducing the upper bounds for this function.

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VI. DIFFERENCES OF POWERS

A. INTRODUCTION

In this document we use the word <u>power</u> to denote a number of the form x^m where x and m are positive integers and $m > 1$. We consider which positive integers z can be expressed as the difference of two powers; that is,

$$z = x^m - y^n, \quad z > 0. \tag{1}$$

We have employed the CDC 6600 computer to find all solutions to (1) in which $z \le 1000$ and $y^n < x^m \le 10^{24}$.

A direct method can be used to solve (1) for $z \le D$ and $y^n < x^m \le L$. For $x > 1$, consider the n^{th} powers nearest to x^m. If u is the greatest integer in $x^{m/n}$, these are u^n, $(u \pm 1)^n$, $(u \pm 2)^n$, ..., . If $h \ge 0$ and $z = x^m - (u - h)^n \le D$, we obtain a solution. If $x^m - 1 \le D$, then each h with $0 \le h \le x - 1$ is an acceptable value. Similarly, for $z = (u + h)^n - x^m \le D$, $h \nmid 0$, we obtain a solution. By examining the successive values h = 0, 1, 2, ..., we obtain all solutions in which the power x^m appears; a simple test indicates where to terminate each sequence. If $n = m$, it is only necessary to consider $z = x^m - (x - h)^m$ for h = 1, 2, 3, ..., as long as $z \le D$, or until $h = x - 1$. In this case an upper limit to x is the least integer X such that $(X + 1)^m - X^m > D$. Since for $m = pq$ and $n = rs$, $x^m - y^n = (x^p)^q - (y^r)^s$, if the foregoing procedure is carried out for all values of x with $x^m \le L$, then only prime exponents m and n, with $n \le m$, need be examined. For $L = 10^{24}$ and $D = 1000$, it is necessary to consider 500 squares (since $501^2 - 500^2 > 1000$), $10^{24/3} = 10^8$ cubes, $63095 = [10^{24/5}]$ fifth powers, and so forth. The largest exponent m tried was 79, because $2^{80} > 10^{24}$.

Since the number of cubes to be examined in solving $z = \pm(x^3 - y^2)$ is so large, a special method was employed. (This was desirable especially since the larger values of x^3 and y^2 are integers of up to 24 digits, necessitating the use of double precision arithmetic.) If x is large compared to D, then $x^{3/2}$ must be very nearly an integer. For, setting $x^{3/2} = y + a$ where y is an integer and $|a| < 1$, we have

$$D \geq |x^3 - y^2| = |(y + a)^2 - y^2| = |a(2y + a)| > |a|(2y - 1),$$

so that $|a| < \frac{D}{2y - 1} < \frac{D}{2x^{3/2} - 3}$. If $x > 90,000$ and $D = 1000$, $|a| < 2 \times 10^{-5}$. We can therefore require that the fractional part of $x^{3/2}$ be very close to 0 or 1 before proceeding to compute the difference between x^3 and the squares nearest to x^3.

Now let $x = t^2 + u$ where t, u are integers, $t > 0$ and $1 \leq u \leq 2t$. Then

$$x^{3/2} = t^3\left(1 + \frac{u}{t^2}\right)^{3/2} = t^3 + \frac{3tu}{2} + \frac{3u^2}{8t} - \frac{u^3}{16t^3} + \frac{3u^4}{128t^5} - \frac{3u^5}{256t^7} + \ldots \tag{2}$$

If $\lambda(t, u) = 0$ when tu is even and $= 1/2$ when tu is odd, the fractional part of $x^{3/2}$ is the same as the fractional part of $f(t, u) = \lambda(t, u) + \frac{3u^2}{8t} - \frac{u^3}{16t^3}$ with an error not exceeding $\frac{3u^4}{128t^5} \leq \frac{3}{128}\,\frac{16t^4}{t^5} = \frac{3}{8t}$. If $x > 90,000$, then $t > 300$ and this error is < 0.00125. The direct method was used for $x \leq 90,000$ and, for $90,000 < x \leq 10^8$, each x was rejected unless the fractional part of f(t, u) fell either in the range (0, 0.0013) or (0.9987, 1), in which case the direct method was applied to x.

The power series expansion for $x^{3/2}$, (2), leads to the following sets of polynomials x, y, z such that $z = x^3 - y^2$ is (relatively) small.

$$x = t^2 + u,\ y = t^3 + \frac{3tu}{2},\ z = \frac{3}{4} t^2 u^2 + u^3, \text{ where } tu \text{ is even;} \tag{3}$$

$$\begin{aligned} t &= 6p^2/d \text{ where } d \text{ divides } 6p^2 \\ u &= 4p \\ x &= t^2 - u,\ y = t^3 - \frac{3}{2} tu + d,\ z = 8p^3 - d^2. \end{aligned} \tag{4}$$

For example, (3) gives for $u = 1$, $t = 2$, $x = 5$, $y = 11$, $z = x^3 - y^2 = 4$, and (4) gives for $d = 1$, $p = 5$, $x = 22480$, $y = 3370501$, $z = x^3 - y^2 = 999$.

B. RESULTS

There are 2895 solutions to (1) with $1 \le z \le 1000$ and $y^n < x^m \le 10^{24}$. Of these, 274 solutions have $z \le 100$ and these are given in Table XVI. Some characteristics of the larger set of solutions are given in the following discussion.

Let G(t) denote the number of solutions to (1) for $0 < z \le t$. S. Pillai conjectured [1] that G(t) is finite for all $t > 0$. We give in Table XVII a tabulation of F(t), a lower bound to G(t), obtained by counting the solutions for which $x^m \le 10^{24}$. F(t) is approximately linear in the interval $0 < t \le 1000$, and for $100 \le t \le 1000$ and we have $2.7 < F(t)/t < 2.9$.

To suggest that G(t) is indeed finite and is closely approximated by F(t) we give in Table XVIII the number of solutions to (1) with $z \le 1000$ in which x^m has k digits, for $k = 1, 2, 3, \ldots, 18$. (There are no solutions with $19 \le k \le 24$.) It seems likely that the total number of solutions for $z \le 1000$ in which x^m exceeds 10^{24} is a small integer or zero.

Table XVI

Solutions of $z = x^m - y^n$ for $0 < z \le 1000$, $y^n < x^m \le 10^{24}$

z	$x^m - y^n$	
1	$3^2 - 2^3$	9 - 8
2	$3^3 - 5^2$	27 - 25
3	$2^2 - 1$	4 - 1
	$2^7 - 5^3$	128 - 125
4	$2^3 - 2^2$	8 - 4
	$6^2 - 2^5$	36 - 32
	$5^3 - 11^2$	125 - 121
5	$3^2 - 2^2$	9 - 4
	$2^5 - 3^3$	32 - 27
6	none	
7	$2^3 - 1$	8 - 1
	$2^4 - 3^2$	16 - 9
	$2^5 - 5^2$	32 - 25
	$2^7 - 11^2$	128 - 121
	$2^{15} - 181^2$	32768 - 32761
8	$3^2 - 1$	9 - 1
	$2^4 - 2^3$	16 - 8
	$312^2 - 46^3$	97344 - 97336
9	$5^2 - 2^4$	25 - 16
	$6^2 - 3^3$	36 - 27
	$15^2 - 6^3$	225 - 216
	$253^2 - 40^3$	64009 - 64000
10	$13^3 - 3^7$	2197 - 2187

z	$x^m - y^n$	
11	$3^3 - 2^4$	27 - 16
	$6^2 - 5^2$	36 - 25
	$56^2 - 5^5$	3136 - 3125
	$15^3 - 58^2$	3375 - 3364
12	$2^4 - 2^2$	16 - 4
	$47^2 - 13^3$	2209 - 2197
13	$7^2 - 6^2$	49 - 36
	$2^8 - 3^5$	256 - 243
	$17^3 - 70^2$	4913 - 4900
14	none	
15	$2^4 - 1$	16 - 1
	$2^6 - 7^2$	64 - 49
	$138^2 - 109^3$	1295044 -1295029
16	$5^2 - 3^2$	25 - 9
	$2^5 - 2^4$	32 - 16
	$12^2 - 2^7$	144 - 128
17	$5^2 - 2^3$	25 - 8
	$7^2 - 2^5$	49 - 32
	$3^4 - 2^6$	81 - 64
	$23^2 - 2^9$	529 - 512
	$282^2 - 43^3$	79524 - 79507
	$375^2 - 52^3$	140625 - 140608
	$378661^2 - 5234^3$	143384152921 -143384152904

Table XVI

Solutions of $z = x^m - y^n$ for $0 < z \leq 1000$, $y^n < x^m \leq 10^{24}$ (Continued)

z	$x^m - y^n$	
18	$3^3 - 3^2$	27 - 9
	$3^5 - 15^2$	243 - 225
	$19^2 - 7^3$	361 - 343
19	$3^3 - 2^3$	27 - 8
	$10^2 - 3^4$	100 - 81
	$12^2 - 5^3$	144 - 125
	$7^3 - 18^2$	343 - 324
	$55^5 - 22434^2$	503284375 -503284356
20	$6^2 - 2^4$	36 - 16
	$6^3 - 14^2$	216 - 196
21	$5^2 - 2^2$	25 - 4
	$11^2 - 10^2$	121 - 100
22	$7^2 - 3^3$	49 - 27
	$47^2 - 3^7$	2209 - 2187
23	$3^3 - 2^2$	27 - 4
	$2^5 - 3^2$	32 - 9
	$12^2 - 11^2$	144 - 121
	$2^{11} - 45^2$	2048 - 2025
24	$5^2 - 1$	25 - 1
	$2^5 - 2^3$	32 - 8
	$7^2 - 5^2$	49 - 25
	$2^{10} - 10^3$	1024 - 1000
	$736844^2 - 8158^3$	542939080336 -542939080312
25	$5^3 - 10^2$	125 - 100
	$13^2 - 12^2$	169 - 144
26	$3^3 - 1$	27 - 1
	$35^3 - 207^2$	42875 - 42849
	$537^2 - 23^5$	6436369 -6436343
27	$6^2 - 3^2$	36 - 9
	$14^2 - 13^2$	196 - 169
	$3^5 - 6^3$	243 - 216
28	$2^5 - 2^2$	32 - 4
	$6^2 - 2^3$	36 - 8
	$2^6 - 6^2$	64 - 36
	$2^7 - 10^2$	128 - 100
	$2^9 - 22^2$	512 - 484
	$37^3 - 15^4$	50653 - 50625
	$2^{17} - 362^2$	131072 - 131044
29	$15^2 - 14^2$	225 - 196
30	$83^2 - 19^3$	6889 - 6859
31	$2^5 - 1$	32 - 1
	$2^8 - 15^2$	256 - 225
32	$6^2 - 2^2$	36 - 4
	$2^6 - 2^5$	64 - 32
	$3^4 - 7^2$	81 - 49
	$6^5 - 88^2$	7776 - 7744
33	$7^2 - 2^4$	49 - 16
	$17^2 - 2^8$	289 - 256
34	none	

Table XVI

Solutions of $z = x^m - y^n$ for $0 < z \le 1000$, $y^n < x^m \le 10^{24}$ (Continued)

z	$x^m - y^n$	
35	$6^2 - 1$	36 - 1
	$18^2 - 17^2$	324 - 289
	$11^3 - 6^4$	1331 - 1296
36	$10^2 - 2^6$	100 - 64
	$42^2 - 12^3$	1764 - 1728
37	$2^6 - 3^3$	64 - 27
	$19^2 - 18^2$	361 - 324
	$788^2 - 3^{15}$	14348944 -14348907
38	$37^2 - 11^3$	1369 - 1331
39	$2^6 - 5^2$	64 - 25
	$20^2 - 19^2$	400 - 361
	$10^3 - 31^2$	1000 - 961
	$22^3 - 103^2$	10648 - 10609
40	$7^2 - 3^2$	49 - 9
	$11^2 - 3^4$	121 - 81
	$2^8 - 6^3$	256 - 216
	$14^3 - 52^2$	2744 - 2704
41	$7^2 - 2^3$	49 - 8
	$13^2 - 2^7$	169 - 128
	$21^2 - 20^2$	441 - 400
42	none	
43	$22^2 - 21^2$	484 - 441
44	$5^3 - 3^4$	125 - 81
	$12^2 - 10^2$	144 - 100
	$13^2 - 5^3$	169 - 125
45	$7^2 - 2^2$	49 - 4
	$3^4 - 6^2$	81 - 36
	$23^2 - 22^2$	529 - 484
	$21^3 - 96^2$	9261 - 9216
46	$17^2 - 3^5$	289 - 243
47	$2^7 - 3^4$	128 - 81
	$6^3 - 13^2$	216 - 169
	$3^5 - 14^2$	243 - 196
	$24^2 - 23^2$	576 - 529
	$12^3 - 41^2$	1728 - 1681
	$63^3 - 500^2$	250047 - 250000
48	$7^2 - 1$	49 - 1
	$2^6 - 2^4$	64 - 16
	$13^2 - 11^2$	169 - 121
	$28^3 - 148^2$	21952 - 21904
49	$3^4 - 2^5$	81 - 32
	$5^4 - 24^2$	625 - 576
	$65^3 - 524^2$	274625 - 274576
50	none	
51	$10^2 - 7^2$	100 - 49
	$26^2 - 5^4$	676 - 625
52	$14^2 - 12^2$	196 - 144
53	$3^6 - 26^2$	729 - 676
	$29^3 - 156^2$	24389 - 24336

Table XVI

Solutions of $z = x^m - y^n$ for $0 < z \leq 1000$, $y^n < x^m \leq 10^{24}$ (Continued)

z	$x^m - y^n$	
54	$3^4 - 3^3$	81 - 27
	$7^3 - 17^2$	343 - 289
55	$2^6 - 3^2$	64 - 9
	$28^2 - 3^6$	784 - 729
	$56^3 - 419^2$	175616 - 175561
56	$2^6 - 2^3$	64 - 8
	$3^4 - 5^2$	81 - 25
	$15^2 - 13^2$	225 - 169
	$18^3 - 76^2$	5832 - 5776
57	$11^2 - 2^6$	121 - 64
	$20^2 - 7^3$	400 - 343
	$29^2 - 28^2$	841 - 784
58	none	
59	$30^2 - 29^2$	900 - 841
60	$2^6 - 2^2$	64 - 4
	$2^8 - 14^2$	256 - 196
	$136^3 - 1586^2$	2515456 -2515396
	$76^5 - 50354^2$	2535525376 -2535525316
61	$5^3 - 2^6$	125 - 64
	$31^2 - 30^2$	961 - 900
62	none	
63	$2^6 - 1$	64 - 1
	$12^2 - 3^4$	144 - 81
	$2^{10} - 31^2$	1024 - 961
	$568^3 - 13537^2$	183250432 -183250369
64	$10^2 - 6^2$	100 - 36
	$2^7 - 2^6$	128 - 64
	$17^2 - 15^2$	289 - 225
	$24^2 - 2^9$	576 - 512
65	$3^4 - 2^4$	81 - 16
	$33^2 - 2^{10}$	1089 - 1024
	$53^2 - 14^3$	2809 - 2744
	$14113^2 - 584^3$	199176769 -199176704
66	none	
67	$34^2 - 33^2$	1156 - 1089
	$23^3 - 110^2$	12167 - 12100
68	$10^2 - 2^5$	100 - 32
	$14^2 - 2^7$	196 - 128
	$18^2 - 2^8$	324 - 256
	$46^2 - 2^{11}$	2116 - 2048
	$874^2 - 152^3$	3511876 -3511808
69	$13^2 - 10^2$	169 - 100
	$35^2 - 34^2$	1225 - 1156
70	none	
71	$14^2 - 5^3$	196 - 125
	$2^9 - 21^2$	512 - 441
	$6^4 - 35^2$	1296 - 1225
	$3^7 - 46^2$	2187 - 2116

Table XVI

Solutions of $z = x^m - y^n$ for $0 < z \le 1000$, $y^n < x^m \le 10^{24}$ (Continued)

z	$x^m - y^n$		z	$x^m - y^n$	
72	$3^4 - 3^2$	81 - 9	80	$3^4 - 1$	81 - 1
	$11^2 - 7^2$	121 - 49		$12^2 - 2^6$	144 - 64
	$6^3 - 12^2$	216 - 144		$21^2 - 19^2$	441 - 361
	$19^2 - 17^2$	361 - 289		$292^2 - 44^3$	85264 - 85184
73	$3^4 - 2^3$	81 - 8	81	$15^2 - 12^2$	225 - 144
	$10^2 - 3^3$	100 - 27		$18^2 - 3^5$	324 - 243
	$17^2 - 6^3$	289 - 216		$41^2 - 40^2$	1681 - 1600
	$37^2 - 6^4$	1369 - 1296		$13^3 - 46^2$	2197 - 2116
	$611^2 - 72^3$	373321 - 373248	82	none	
	$717^2 - 356^3$	45118089 -45118016	83	$42^2 - 41^2$	1764 - 1681
				$3^9 - 140^2$	19683 - 19600
74	$3^5 - 13^2$	243 - 169	84	$10^2 - 2^4$	100 - 16
	$99^3 - 985^2$	970299 - 970225		$22^2 - 20^2$	484 - 400
75	$10^2 - 5^2$	100 - 25	85	$11^2 - 6^2$	121 - 36
	$14^2 - 11^2$	196 - 121		$43^2 - 42^2$	1849 - 1764
	$38^2 - 37^2$	1444 - 1369	86	none	
76	$5^3 - 7^2$	125 - 49	87	$2^8 - 13^2$	256 - 169
	$20^2 - 18^2$	400 - 324		$7^3 - 2^8$	343 - 256
	$101^3 - 1015^2$	1030301 -1030225		$44^2 - 43^2$	1936 - 1849
77	$3^4 - 2^2$	81 - 4	88	$13^2 - 3^4$	169 - 81
	$39^2 - 38^2$	1521 - 1444		$6^3 - 2^7$	216 - 128
78	none			$23^2 - 21^2$	529 - 441
79	$2^7 - 7^2$	128 - 49			
	$40^2 - 39^2$	1600 - 1521			
	$20^3 - 89^2$	8000 - 7921			
	$302^2 - 45^3$	91204 - 91125			

Table XVI

Solutions of $z = x^m - y^n$ for $0 < z \le 1000$, $y^n < x^m \le 10^{24}$ (Concluded)

z	$x^m - y^n$	
89	$11^2 - 2^5$	121 - 32
	$5^3 - 6^2$	125 - 36
	$33^2 - 10^3$	1089 - 1000
	$45^2 - 44^2$	2025 - 1936
	$91^2 - 2^{13}$	8281 - 8192
	$408^2 - 55^3$	166464 - 166375
90	none	
91	$10^2 - 3^2$	100 - 9
	$6^3 - 5^3$	216 - 125
	$46^2 - 45^2$	2116 - 2025
92	$10^2 - 2^3$	100 - 8
	$2^7 - 6^2$	128 - 36
	$24^2 - 22^2$	576 - 484
	$2^{13} - 90^2$	8192 - 8100
93	$5^3 - 2^5$	125 - 32
	$17^2 - 14^2$	289 - 196
	$47^2 - 46^2$	2209 - 2116
	$130^2 - 7^5$	16900 - 16807
94	$11^2 - 3^3$	121 - 27
	$421^2 - 3^{11}$	177241 - 177147
95	$12^2 - 7^2$	144 - 49
	$6^3 - 11^2$	216 - 121
	$48^2 - 47^2$	2304 - 2209
	$6^7 - 529^2$	279936 - 279841
96	$10^2 - 2^2$	100 - 4
	$11^2 - 5^2$	121 - 25
	$2^7 - 2^5$	128 - 32
	$14^2 - 10^2$	196 - 100
	$5^4 - 23^2$	625 - 529
97	$15^2 - 2^7$	225 - 128
	$7^4 - 48^2$	2401 - 2304
	$77^2 - 18^3$	5929 - 5832
98	$5^3 - 3^3$	125 - 27
	$21^2 - 7^3$	441 - 343
99	$10^2 - 1$	100 - 1
	$3^5 - 12^2$	243 - 144
	$18^2 - 15^2$	324 - 225
	$50^2 - 7^4$	2500 - 2401
100	$5^3 - 5^2$	125 - 25
	$15^2 - 5^3$	225 - 125
	$7^3 - 3^5$	343 - 243
	$26^2 - 24^2$	676 - 576
	$10^3 - 30^2$	1000 - 900
	$5^5 - 55^2$	3125 - 3025
	$90^2 - 20^3$	8100 - 8000
	$118^2 - 24^3$	13924 - 13824
	$34^3 - 198^2$	39304 - 39204
	$137190^2 - 2660^3$	18821096100 -18821096000

Table XVII

F(t), the Number of Solutions to $z = x^m - y^n$ for $0 < z \le t$ in Which $y^n < x^m \le 10^{24}$

t	100	200	300	400	500	600	700	800	900	1000
F(t)	274	564	855	1136	1413	1710	1991	2287	2598	2895
F(t)/t	2.740	2.820	2.850	2.840	2.826	2.850	2.844	2.859	2.887	2.895

Table XVIII

Number of Solutions, N(k), with k Digits

k	1	2	3	4	5	6	7	8	9
N(k)	6	72	742	963	605	296	72	47	33

k	10	11	12	13	14	15	16	17	18
N(k)	20	16	8	4	3	3	1	2	2

Table XIX

Number of Values R(k) of z for Which There Are k Solutions

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
R(k)	138	124	223	173	158	69	55	25	16	5	8	3	1	2

For each $z \le 1000$ there are at most 13 solutions in the range considered. We give in Table XIX the number of values of z for which there are k solutions, $0 \le k \le 13$. The two values of $z \le 1000$ with the maximum number of representations as the difference of two powers are

$$\begin{aligned} 225 &= 17^2 - 2^6 = 5^4 - 20^2 = 39^2 - 6^4 = 113^2 - 112^2 = 21^2 - 6^3 \\ &= 35^2 - 10^3 = 60^2 - 15^3 = 165^2 - 30^3 = 465^2 - 60^3 = 2415^2 - 180^3 \\ &= 6159^2 - 336^3 = 6576^2 - 351^3 = 611085363^2 - 720114^3; \end{aligned}$$

and

$$\begin{aligned} 775 &= 28^2 - 3^2 = 30^2 - 5^3 = 10^3 - 15^2 = 80^2 - 75^2 = 19^3 - 78^2 = 20^3 - 85^2 \\ &= 264^2 - 41^3 = 10^5 - 315^2 = 388^2 - 387^2 = 70^3 - 585^2 = 80^3 - 715^2 \\ &= 16750^3 - 2167815^2 = 26530^3 - 4321215^2. \end{aligned}$$

The 138 values of z for which (1) has no solutions (in the range covered) are given in Table XX. All these are of the form $4k + 2$, since any other positive integer is the difference of two squares (except $1 = 3^2 - 2^3$ and $4 = 2^3 - 2^2$).

In Table XXI we give the solution to (1) in largest integers, over the range considered, for each pair of prime values m, n. Where there is no entry for a given pair m, n, all positive differences $x^m - y^n$ exceed 1000. If the upper limit on z is increased to 10,000, the only additional solutions in largest integers obtained in which either m or n exceeds 19 are $9796 = 2898^2 - 2^{23}$, $1792 = 2^{23} - 2896^2$ and $4633 = 46341^2 - 2^{31}$. (Note: $z = 4002 = 2897^2 - 2^{23}$ is not in largest integers x, y for a particular pair m, n.)

The equation $\pm z = x^3 - y^2$ was studied on a computer by M. Lal, M. Jones and W. Blundon [2], who reported 8593 solutions in the range $|z| < 10^4$ and $x < 4.64 \times 10^6$. Since negative values for x were allowed, the results

Table XX

Values of z for Which $z = x^m - y^n$ Has No Solution, $x^m \le 10^{24}$

Range	Number of Values	z
1 - 100	13	6 14 34 42 50 58 62 66 70 78 82 86 90
101 - 200	8	102 110 114 130 134 158 178 182
201 - 300	13	202 206 210 226 230 238 246 254 258 266 274 278 290
301 - 400	13	302 306 310 314 322 326 330 358 374 378 390 394 398
401 - 500	20	402 410 418 422 426 430 434 438 442 446 450 454 458 462 466 470 474 478 482 494
501 - 600	14	510 514 522 526 530 534 538 542 554 558 562 570 578 590
601 - 700	16	606 610 614 626 630 634 642 646 650 654 658 662 670 682 690 698
701 - 800	14	714 722 738 742 750 754 758 762 770 778 786 790 794 798
801 - 900	12	810 822 826 830 842 854 858 862 870 874 886 898
901 - 1000	15	902 910 918 922 926 942 946 950 958 966 978 986 990 994 998
1 - 1000	138	

Table XXI

Solutions to $z = x^m - y^n$ in Largest Integers x^m, y^n for $0 < z \le 1000$ and $y^n < x^m \le 10^{24}$ m, n Prime

m	n	x	y	z
2	2	500	499	999
3	2	939 787	911 054 064	307
2	3	611 085 363	720 114	225
3	3	18	17	919
5	2	377	2 759 646	341
5	3	6	19	917
2	5	45 531	73	368
3	5	39	9	270
5	5	4	3	781
7	2	8	1 448	448
7	3	4	25	759
7	5	2	2	96
2	7	5 986	12	388
3	7	14	3	557
5	7	7	4	423
7	7	2	1	128
11	2	3	420	747
11	3	2	12	320
2	11	422	3	937
3	11	14	2	696
7	11	3	2	139
13	2	2	85	967
13	3	2	20	192
13	5	2	6	416
2	13	1 263	3	846
17	2	2	361	751
2	17	11 364	3	333
19	2	2	724	112

are not exactly comparable to those given here. In our calculations we recorded differences of powers up to $z \le 10,000$ and can state that for $4.64 \times 10^6 \le x \le 10^8$ there are only four additional solutions:

$$-3753 = 5024238^3 - 11261735055^2$$
$$7670 = 6369039^3 - 16073515093^2$$
$$-6856 = 27564105^3 - 144715764559^2$$
$$-1090 = 28187351^3 - 149651610621^2$$

Catalan's conjecture, that equation (1) with $z = 1$ has only one non-trivial solution, is a problem studied much but not yet completely solved. J. W. S. Cassels [3] and K. Inkeri [4] have made recent contributions to the theory of this problem.

C. ARITHMETIC PROGRESSIONS

If $z = x^m - y^n = y^n - w^k$, then the three powers w^k, y^n, x^m form an arithmetic progression with common difference z. The set of solutions to (1) was examined to find values of z for which there were two solutions (or more) having a common term. There are thirty-one progressions of three terms identified in this way, listed in Table XXII. There are no progressions of four terms in the range covered.

A. Guibert [5] gave the general solution for three relatively prime squares a^2, b^2, c^2 in arithmetic progression:

$$a = \pm (p^2 - q^2 - 2pq), \ b = p^2 + q^2, \ c = p^2 - q^2 + 2pq,$$

where p and q are relatively prime and not both are odd. Euler [6] proved that four distinct squares cannot be in arithmetic progression.

Table XXII

Arithmetic Progressions

Difference	Progression	Powers
24	1 25 49	1 5^2 7^2
28	8 36 64	2^3 6^2 2^6
44	81 125 169	3^4 5^3 13^2
87	169 256 343	13^2 2^8 7^3
96	4 100 196	2^2 10^2 14^2
100	25 125 225	5^2 5^3 15^2
112	32 144 256	2^5 12^2 2^8
118	125 243 361	5^3 3^5 19^2
120	49 169 289	7^2 13^2 17^2
141	343 484 625	7^3 22^2 5^4
147	49 196 343	7^2 14^2 7^3
184	32 216 400	2^5 6^3 20^2
216	9 225 441	3^2 15^2 21^2
240	49 289 529	7^2 17^2 23^2
336	289 625 961	17^2 5^4 31^2
384	16 400 784	2^4 20^2 28^2
433	1331 1764 2197	11^3 42^2 13^3
441	343 784 1225	7^3 28^2 35^2
448	128 576 1024	2^7 24^2 2^{10}
480	196 676 1156	14^2 26^2 34^2
600	25 625 1225	5^2 5^4 35^2
600	400 1000 1600	20^2 10^3 40^2
720	961 1681 2401	31^2 41^2 7^4

Table XXII (Concluded)

Arithmetic Progressions

Difference	Progression	Powers
828	1369 2197 3025	37^2 13^3 55^2
840	1 841 1681	1 29^2 41^2
840	529 1369 2209	23^2 37^2 47^2
864	36 900 1764	6^2 30^2 42^2
936	64 1000 1936	2^6 10^3 44^2
944	3025 3969 4913	55^2 63^2 17^3
960	196 1156 2116	14^2 34^2 46^2
980	784 1764 2744	28^2 42^2 14^3

There are infinitely many progressions of the form a^2, b^n, c^2 where $n > 2$. For, if b is the sum of two squares, so is $2b^n = a^2 + c^2$. The smallest examples are 81, 125, 169 and 25, 125, 225 derived from $2 \cdot 5^3 = 9^2 + 13^2 = 5^2 + 15^2$.

It has been shown that three distinct positive cubes [7] or biquadrates [8] cannot be in arithmetic progression. There are no known instances of three distinct positive n^{th} powers in arithmetic progression for $n \geq 5$. That is, $a^n + b^n = 2c^n$ has no known non-trivial solutions for $n \geq 3$. A. Makowski [9] proved that there is no progression of three powers with the common difference 1.

It is possible to construct any number of arithmetic progressions of arbitrary length in which each term is a power. To show this, let $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ be a progression of powers with $r \geq 2$ terms and common difference $d \neq 0$. Then, if $y = x_r^{a_r} + d$ is not already a power, let c be the least common multiple of $a_1, a_2, \ldots, a_r$ and let $c = a_i b_i$ for $1 \leq i \leq r$. Then the sequence

$$y^c x_1^{a_1},\ y^c x_2^{a_2},\ \ldots,\ y^c x_r^{a_r},\ y^{c+1}$$

is the same as

$$(y^{b_1} x_1)^{a_1},\ (y^{b_2} x_2)^{a_2},\ \ldots,\ (y^{b_r} x_r)^{a_r},\ y^{c+1},$$

a progression of powers with $r + 1$ terms and common difference $y^c d$. For example, from the progression $8, 36, 64 = 2^3, 6^2, 2^6$ whose next term is $64 + 28 = 92$, we can derive the progression

$$92^6 \cdot 2^3,\ 92^6 \cdot 6^2,\ 92^6 \cdot 2^6,\ 92^7 = (2 \cdot 92^2)^3,\ (6 \cdot 92^3)^2,\ (2 \cdot 92)^6,\ 92^7$$

of four terms. This method produces progressions in which all terms have a common factor > 1. It is not known whether there exists a progression of powers of more than three terms for which no integer > 1 divides every term. We have seen that if there is such a progression, the common difference must exceed 1000 or one of the terms must exceed 10^{24}.

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VII. TWO EQUAL SUMS OF FOUR FIFTH POWERS

A. DISCUSSION

We shall derive three parametric solutions of the Diophantine equation

$$A_1^5 + A_2^5 + A_3^5 + A_4^5 = B_1^5 + B_2^5 + B_3^5 + B_4^5 \tag{1}$$

for which we also use the notation $(A_1, A_2, A_3, A_4)^5 = (B_1, B_2, B_3, B_4)^5$. By generating and sorting numbers of the form $A_1^5 + A_2^5 + A_3^5 + A_4^5$ on a digital computer it was observed that (1) has many solutions in moderately small positive integers and that a large proportion of these solutions satisfy the additional conditions

$$A_1 + A_2 = B_1 + B_2 \qquad A_3 + A_4 = B_3 + B_4. \tag{2}$$

For example, the solution of (1) in least integers is $(5, 6, 6, 8)^5 = (4, 7, 7, 7)^5$, which also satisfies (2), since $5 + 6 = 4 + 7$, $6 + 8 = 7 + 7$.

If we set

$$\begin{aligned} A_1 &= u_1 + v_1 + w_1 & B_1 &= u_1 + v_1 - w_1 \\ A_2 &= u_1 - v_1 - w_1 & B_2 &= u_1 - v_1 + w_1 \\ A_3 &= u_2 + v_2 - w_2 & B_3 &= u_2 + v_2 + w_2 \\ A_4 &= u_2 - v_2 + w_2 & B_4 &= u_2 - v_2 - w_2 \end{aligned} \tag{3}$$

it follows at once that (3) is a solution of (2). Equation (1) will also be satisfied provided that

$$u_1 v_1 w_1 (u_1^2 + v_1^2 + w_1^2) = u_2 v_2 w_2 (u_2^2 + v_2^2 + w_2^2). \tag{4}$$

For the least solution, $(u_1, v_1, w_1) = (11, 2, 1)$, $(u_2, v_2, w_2) = (14, 1, 1)$, and $11 \cdot 2 \cdot 1\,(11^2 + 2^2 + 1^2) = 22 \cdot 126 = 2 \cdot 11 \cdot 9 \cdot 14 = 14 \cdot 198 = 14 \cdot 1 \cdot 1\,(14^2 + 1^2 + 1^2)$.

The first two parametric solutions are obtained by applying to (4) a well-known geometric method related to plane curves of the third degree. If P is a rational point on a curve C, and if the tangent to C at P intersects C again, the intersection has rational coordinates. If C has an asymptote A with rational slope and the line through P parallel to A intersects C again, this intersection is also a rational point.

Consider the Cartesian curve C with equation

$$xab\,(x^2 + a^2 + b^2) = yac\,(y^2 + a^2 + c^2) \qquad abc \neq 0, \tag{5}$$

which has an asymptote $bx^3 = cy^3$ of rational slope m, provided $b = cm^3$. Then $x = c$, $y = b$ is trivially a rational point P on C. The line through P with slope m intersects C again in the point with coordinates

$$x = [a^2 + c^2 + c^2m^2\,(m^2 - 1)^2]/3m^2c$$

$$y = mx + mc\,(m^2 - 1).$$

After clearing of fractions, this gives the following parametric solution to (4):

$$\begin{aligned} u_1 &= a^2 + (m^6 - 2m^4 + m^2 + 1)\,c^2 \\ v_1 &= 3m^5\,c^2 \\ w_1 &= 3m^2\,ac \\ u_2 &= ma^2 + (m^7 + m^5 - 2m^3 + m)\,c^2 \\ v_2 &= 3m^2\,c^2 \\ w_2 &= 3m^2\,ac. \end{aligned} \tag{6}$$

For each rational m there results a solution to (1) in homogeneous polynomials of the second degree in two variables. The least primitive solutions to (1) produced by (6) with integer arguments have

$(m, a, c) = (2, 1, 1)$ and $(2, 1, 3)$ which give $(23, 73, 74, 74)^5 = (35, 61, 62, 86)^5$ and $(7, 70, 89, 94)^5 = (43, 53, 58, 106)^5$, respectively.

Next, the tangent to C at P has the equation

$$y = k_1 x + k_2 \tag{7}$$

where $k_1 = \dfrac{b\,(a^2 + b^2 + 3c^2)}{c\,(a^2 + 3b^2 + c^2)}$, $k_2 = b - k_1 c$

and intersects C again in a point with abscissa

$$x = \frac{3c\, k_1^2\, k_2}{b - ck_1^3} - 2c\,. \tag{8}$$

Equations (7) and (8) together with

$$\begin{array}{lll} u_1 = x & v_1 = b & w_1 = a \\ u_2 = y & v_2 = c & w_2 = a \end{array} \tag{9}$$

provide another solution to (4) in the three parameters a, b, c. The two least primitive solutions to (1) produced by (9) have $(a, b, c) = (2, 1, 5)$ and $(2, 1, 3)$, giving respectively $(11, 17, 22, 28)^5 = (7, 21, 24, 26)^5$ and $(28, 30, 39, 45)^5 = (24, 34, 41, 43)^5$.

A third solution is derived by taking (4) together with the additional condition

$$u_1 v_1 w_1 = u_2 v_2 w_2\,, \tag{10}$$

which implies (for non-trivial solutions)

$$u_1^2 + v_1^2 + w_1^2 = u_2^2 + v_2^2 + w_2^2\,.$$

(When (3), (10), and (11) are satisfied, $(A_1, A_2, A_3, A_4)^n = (B_1, B_2, B_3, B_4)^n$ for n = 1 3 and 5.)

A solution of (11) can be written

$$u_1 = x + a \qquad v_1 = y - b \qquad w_1 = z - c$$
$$u_2 = x - a \qquad v_2 = y + b \qquad w_2 = z + c \tag{12}$$

where a, b, c, x, y, z are any rational numbers satisfying the single condition

$$ax = by + cz. \tag{13}$$

Substituting (12) into (10) and using (13) to eliminate x gives

$$(yz)\, b^2 + c(y^2 + z^2 - a^2)\, b + yz(c^2 - a^2) = 0. \tag{14}$$

If b is rational, the discriminant of this quadratic equation must be a square, say k^2. This condition may be written

$$c^2\, (y + a + z)\, (y + a - z)\, (y - a + z)\, (y - a - z) = (k + 2ayz)\, (k - 2ayz). \tag{15}$$

Assuming that

$$k + 2ayz = c\, (y + a + z)\, (y - a + z)$$
$$k - 2ayz = c\, (y + a - z)\, (y - a - z)\,, \tag{16}$$

we find using (14) and (13) that

$$a = c,\ k = c\, (y^2 + z^2 - c^2),\ b = -\frac{c}{yz}\, (y^2 + z^2 - c^2),$$
$$x = \frac{c^2 - y^2}{z}\ . \tag{17}$$

After clearing of fractions and making a slight change in notation, the resulting solution to (4) is

$$u_1 = b\, (c^2 - b^2 + ac) \qquad u_2 = b\, (b^2 - c^2 + ac)$$
$$v_1 = (a + c)\, (b^2 - c^2 + ac) \qquad v_2 = (a - c)\, (c^2 - b^2 + ac) \tag{18}$$
$$w_1 = ab\, (a - c) \qquad w_2 = ab\, (a + c)\ .$$

The primitive solutions to (1) in integers not exceeding 50 produced by (18) are given in Table XXIII. In preparing this table terms were transposed so as to be written positively, and common factors were removed.

The following two-parameter solution to (1) was given by G. Xeroudakes and A. Moessner [1]:

$$\begin{aligned}
A_1 &= -p^2 + 4pq + 9q^2 & A_2 &= 3p^2 + 12pq + 21q^2 \\
A_3 &= p^2 + 8pq + 3q^2 & A_4 &= 3p^2 + 12pq + 21q^2 \\
B_1 &= 3p^2 + 16pq + 17q^2 & B_2 &= -p^2 + 13q^2 \\
B_3 &= p^2 + 12pq + 23q^2 & B_4 &= 3p^2 + 8pq + q^2.
\end{aligned} \tag{19}$$

This can be obtained from (18) by setting $a = p + 2q$, $b = 2q$, $c = q$. The first solution in Table I corresponds to this case for $p = -5$, $q = 3$.

Equations (18) produce solutions to (1) which can be written with at most one term non-positive. Two primitive solutions involving a zero term are obtained by setting $(a, b, c) = (1, 10, 14)$ and $(9, 20, 5)$ which give $(38, 105, 123)^5 = (13, 23, 110, 120)^5$ and $(30, 40, 55)^5 = (6, 19, 49, 51)^5$, respectively.

In conclusion we remark that the general solution of (4) is yet to be obtained; none of the parametric solutions contained here produce, for example, the least solution

$$11 \cdot 2 \cdot 1\ (11^2 + 2^2 + 1^2) = 14 \cdot 1 \cdot 1\ (14^2 + 1^2 + 1^2)$$

mentioned previously. The general solution of (1) without the rest iction imposed by (2) is also wanting. The least solution of (1) which does not satisfy (2) was found on the computer to be $(1, 1, 3, 24)^5 = (6, 15, 15, 23)^5$.

Table XXIII

Primitive Solutions to $\sum_1^4 A_i^5 = \sum_1^4 B_i^5$

	a	b	c	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4
1	1	6	3	1	13	17	23	3	9	21	21
2	1	6	2	6	16	18	24	7	13	21	23
3	1	1	3	7	21	23	33	11	13	29	31
4	3	14	10	5	20	22	34	9	12	27	33
5	1	2	4	1	21	27	39	7	11	33	37
6	1	2	5	9	21	31	39	13	15	35	37
7	3	20	7	11	28	34	44	13	23	39	42
8	1	12	5	12	23	39	45	14	20	42	43
9	1	12	3	21	30	39	45	22	28	41	44

B. REFERENCE

[1] G. Xeroudakes and A. Moessner, "On Equal Sums of Like Powers," Proc. Indian Acad. Sci. Sect. A, v. 48, 1958, pp. 245-255.

VIII. EQUAL SUMS OF THREE FIFTH POWERS

A. DISCUSSION

The Diophantine equation

$$A_1^5 + A_2^5 + A_3^5 = B_1^5 + B_2^5 + B_3^5 \qquad (1)$$

was solved in positive integers first by A. Moessner [1] who gave $49^5 + 75^5 + 107^5 = 39^5 + 92^5 + 100^5$. Two parametric solutions were found by Swinnerton-Dyer [2], which result in polynomial solutions of a rather high degree that were not explicitly given. (Dyer did give a single numerical result which appears to be incorrect.) Both Moessner's and Dyer's solutions satisfy (1) together with the additional condition

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 . \qquad (2)$$

In a computer study [3] the solution of (1) in least integers was found to be $24^5 + 28^5 + 67^5 = 3^5 + 54^5 + 62^5$, which also satisfies (2), as do a great many of the other solutions obtained by search. The least solution of (1) which does not also satisfy (2) is $26^5 + 85^5 + 118^5 = 53^5 + 90^5 + 116^5$, showing that (2) is not necessary in solving (1). In this paper we derive by geometric methods a 3-parameter solution to the system (1), (2), which yields Moessner's numbers as well as other solutions in moderately small integers. A particular version of this solution is given explicitly as a set of polynomials of the ninth degree in a single variable.

Suppose that the Diophantine system (1), (2) has the known solution $A_i = a_i$, $B_i = b_i$, $i = 1, 2, 3$. If $u = b_3 - a_3$, $v = a_2 - b_2$, $x_o = a_1 - u$, $y_o = b_2$, $z_o = a_3$; then

$$(x + u)^5 + (y + v)^5 + z^5 = (x + v)^5 + y^5 + (z + u)^5 \qquad (3)$$

is the Cartesian equation of a surface S which passes through the point $P_o\,(x_o, y_o, z_o)$ and contains the parallel lines

$$L_1 : x = y = z , \quad L_2 : x = y - u = z - v .$$

Each rational point $P(x, y, z)$ of S gives a rational solution of (1), (2) through the equations

$$A_1 = x + u \quad A_2 = y + v \quad A_3 = z \quad B_1 = x + v \quad B_2 = y \quad B_3 = z + u . \qquad (4)$$

A solution in integers may be obtained by multiplying A_1, A_2, ..., B_3 so as to clear of fractions.

In expanding (3) the terms in x^5, y^5, z^5 vanish and therefore S is a surface of the fourth degree. A rational line passing through three rational points of S will intersect S again in a fourth rational point. Let T be the plane tangent to S at P_o and P_j the intersection of T with the line L_j, j = 1 or 2. The equation of T is rational, P_j is rational, and thus P_oP_j is a rational line intersecting S in three rational points; that is, twice at P_o and once at P_j. Hence P_oP_j intersects S in a fourth rational point Q_j, which in some cases gives a new solution to (1). If P_o falls on L_1 (L_2), the solution degenerates, because then T contains L_1 (L_2) and is parallel to L_2 (L_1). It may also happen that the new solution is trivial. However, for certain initial solutions a_i, b_i new non-trivial solutions are in fact obtained.

The normal to S at P_o has directions $d_1 : d_2 : d_3$ where $d_i = a_i^4 - b_i^4$ and the coordinates of $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ may be written

$$x_1 = y_1 = z_1 = (x_o d_1 + y_o d_2 + z_o d_3)/(d_1 + d_2 + d_3)$$

$$x_2 = y_2 - u = z_2 - v = x_1 - (d_2 u + d_3 v)/(d_1 + d_2 + d_3) .$$

The equation of P_oP_j can be written parametrically as

$$x = x_o + h_1 t \qquad y = y_o + h_2 t \qquad z = z_o + h_3 t \tag{5}$$

for j = 1 or 2, where $h_1 = x_j - x_o$, $h_2 = y_j - y_o$, $h_3 = z_j - z_o$. On substituting (5) into (3) there results an equation of the form

$$c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_o = 0$$

in which

$$c_4 = 5 \sum_{i=1}^{3} h_i^4 (a_i - b_i) \qquad c_3 = 10 \sum_{i=1}^{3} h_i^3 (a_i^2 - b_i^2) . \tag{6}$$

Three of the roots of this equation are t = 0, 0, 1 and so the fourth root is $t = -(c_3/c_4) - 1$, which together with (5) gives the coordinates of Q_j.

If the minimal solution of (1) is used as an initial solution, the two resulting new solutions (actually there are several sets corresponding to permutations of the a_i and b_i) involve integers of 17 digits. If instead we attempt to use a trivial initial solution in which the b_i are simply a permutation of the a_i, the new solution is either degenerate or trivial.

However, a trivial initial solution of the type

$$a_1 = p \quad a_2 = -p \quad a_3 = r \quad b_1 = q \quad b_2 = r \quad b_3 = -q \tag{7}$$

does produce new solutions. It is sufficient to use Q_2, to take $r > 0$, $p > 0$, $|q| > p$ and have p, q, r all distinct in magnitude. Some results for small integer values of the arguments p, q, r are given in Table XXIV; it will be noted that Moessner's solution is the smallest obtained.

The foregoing provides an implicit parametric solution of (1), (2) in three integer variables (or two rational variables), but does not give the results explicitly in a form which facilitates the calculation of numerical solutions. Accordingly, a specialized solution was computed for (7) with $p = -2$, $q = 1$, using the case $j = 2$ of (5). After clearing of fractions, the result is

$$\begin{aligned}
A_1 &= -2r^8 + 10r^7 + 20r^6 + 20r^5 + 34r^4 - 10r^3 - 270r^2 - 20r - 682\\
A_2 &= 2r^8 + 10r^7 - 20r^6 + 20r^5 - 34r^4 - 10r^3 + 270r^2 - 20r + 682\\
A_3 &= r^9 - 22r^5 - 125r^3 - 79r\\
B_1 &= r^8 + 10r^7 - 10r^6 + 20r^5 - 92r^4 - 160r^3 - 15r^2 - 320r + 341\\
B_2 &= r^9 - 22r^5 + 175r^3 + 521r\\
B_3 &= -r^8 + 10r^7 + 10r^6 + 20r^5 + 92r^4 - 160r^3 + 15r^2 - 320r - 341
\end{aligned} \tag{8}$$

The values of these polynomials for several small rational arguments (after clearing of fractions and removing any common factor) are presented in Table XXV.

In conclusion we note that the solutions obtained can be put in one of the two forms $A^5 + B^5 + C^5 = D^5 + E^5 + F^5$ or $A^5 + B^5 = C^5 + D^5 + E^5 + F^5$ where A, B, C, D, E, F are positive integers. The last three entries in Table I are examples of the latter type.

Table XXIV

Primitive Solutions of $A_1^5 + A_2^5 + A_3^5 = B_1^5 + B_2^5 + B_3^5$

p	q	r	A_1	A_2	A_3	B_1	B_2	B_3
1	-3	2	-907	-549	1378	-1087	-414	1423
1	-2	3	-49	-107	-75	-100	-39	-92
1	2	3	803	289	561	808	309	536
1	3	2	293	-501	754	959	498	-911
1	3	5	1157	543	1135	1271	885	679
2	-4	1	346	-1162	641	-1724	-127	1676
2	3	1	1084	-252	-433	1249	243	-1093
3	4	1	1783	-763	-699	1974	1	-1654

Table XXV

Primitive Solutions of $A_1^5 + A_2^5 + A_3^5 = B_1^5 + B_2^5 + B_3^5$

Derived from a Polynomial Solution

r	A_1	A_2	A_3	B_1	B_2	B_3
3	100	92	39	49	75	107
4	614	1018	1028	773	1124	763
1/2	-1724	1676	-127	346	641	-1162
3/2	-13964	24572	-13701	-19334	25467	-9226
5/2	44684	36516	-2505	2558	29495	46642
8	-2654	20126	53912	14271	53976	3137
2/3	-57738	55866	-6626	4137	28366	-41001
1/4	-74552	73464	-2301	27244	14083	-44716

B. REFERENCES

1. A. Moessner, "Einige Numerische Identitaten," Proc. Indian Acad. Sci., Sect. A, v. 10, 1939, pp. 296-306.

2. H. P. F. Swinnerton-Dyer, "A Solution of $A^5 + B^5 + C^5 = D^5 + E^5 + F^5$," Proc. Cambridge Philos. Soc., v. 48, 1952, pp. 516-518.

3. L. J. Lander, T. R. Parkin and J. L. Selfridge, "A Survey of Equal Sums of Like Powers," Math. Comp., v. 21, 1967.

IX. EQUAL SUMS OF BIQUADRATES

A. DISCUSSION

Solutions of the Diophantine equation $A^4+B^4=C^4+D^4$ in least integers have been obtained by several authors (References 1, 2, 3, 4). The term primitive denotes a solution for which unity is the greatest common divisor of all the numbers A, B, C, D. A CDC 3200 computer program was written to search exhaustively for primitives, yielding the 31 solutions listed in Table XXVI. The range covered is $A^4+B^4<7.885\times10^{15}$. The first six solutions were identified in Reference 3 and the seventh is cited in Reference 1.

Euler (Reference 1) gave a two-parameter algebraic solution which can be written

$$A=f(x,y) \quad B=f(y,-x) \quad C=f(-x,y) \quad D=f(y,x)$$

where $f(x,y)=2x^7-x^6y+20x^5y^2+17x^4y^3+2x^3y^4+17x^2y^5+8xy^6-y^7$. The primitives corresponding to i=1, 7, and 14 of Table XXVI are special cases of this solution for the arguments (x, y)=(3, 1), (2, 1), and (5, 1), respectively.

The computer program generated all values of $N=A^4+B^4$ in ascending order by controlling the advance of a series of pairs of values A, B while monitoring N for coincidences. To advance from a given starting value of N, all integers A for which $N/2 \le A^4 \le N$ were considered; for each A a corresponding B was chosen as the largest integer in the range $0 \le B \le A$ for which $A^4+B^4 \le N$. Then the smallest value $A_1^4+B_1^4$ in the set was found and B_1 was advanced if $B_1<A_1$, the lower limit on A was advanced if $B_1=A_1$, and the upper limit on A was advanced if $B_1=0$.

A similar computer program generated sums of three biquadrates $A^4+B^4+C^4$ in ascending order and found the least triple coincidence to be

$$811,538=29^4+17^4+12^4=28^4+21^4+7^4=27^4+23^4+4^4.$$

It was discovered quite by chance (using a computer program which decomposes numbers into sums of biquadrates by trial) that for the N_i of Table XXVI

Table XXVI

Primitive Solutions of $N=A^4+B^4=C^4+D^4$

i	N_i	A	B	C	D
1	635,318,657	158	59	134	133
2	3,262,811,042	239	7	227	157
3	8,657,437,697	292	193	257	256
4	68,899,596,497	502	271	497	298
5	86,409,838,577	542	103	514	359
6	160,961,094,577	631	222	558	503
7	2,094,447,251,857	1203	76	1176	653
8	4,231,525,221,377	1381	878	1342	997
9	26,033,514,998,417	2189	1324	1997	1784
10	37,860,330,087,137	2461	1042	2141	2026
11	61,206,381,799,697	2797	248	2524	2131
12	76,773,963,505,537	2949	1034	2854	1797
13	109,737,827,061,041	3190	1577	2986	2345
14	155,974,778,565,937	3494	1623	3351	2338
15	156,700,232,476,402	3537	661	3147	2767
16	621,194,785,437,217	4883	2694	4397	3966
17	652,057,426,144,337	5053	604	5048	1283
18	680,914,892,583,617	4849	3364	4303	4288
19	1,438,141,494,155,441	6140	2027	5461	4840
20	1,919,423,464,573,697	6619	274	5942	5093
21	2,089,568,089,060,657	6761	498	6057	5222
22	2,105,144,161,376,801	6730	2707	6701	3070
23	3,263,864,585,622,562	7557	1259	7269	4661
24	4,063,780,581,008,977	7604	5181	7037	6336
25	6,315,669,699,408,737	8912	1657	7559	7432
26	6,884,827,518,602,786	9109	635	9065	3391
27	7,191,538,859,126,257	9018	4903	8409	6842
28	7,331,928,977,565,937	9253	1104	8972	5403
29	7,362,748,995,747,617	9043	5098	8531	6742
30	7,446,891,977,980,337	9289	1142	9097	4946
31	7,532,132,844,821,777	9316	173	9197	4408

$$N_1+1=635,318,658=159^4+58^4+1^4=134^4+133^4+1^4=154^4+83^4+71^4$$

is the sum of three biquadrates in three distinct ways, and that

$N_3+1=8,657,437,698$ is the sum of three biquadrates in five distinct ways, namely,

$$(296,157,139)\stackrel{4}{=}(293,184,109)\stackrel{4}{=}(292,193,1)\stackrel{4}{=}(271,239,32)\stackrel{4}{=}(257,256,1).$$

B. REFERENCES

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X. GEOMETRIC ASPECTS OF EULER'S DIOPHANTINE EQUATION $A^4 + B^4 = C^4 + D^4$

A. INTRODUCTION

Euler [1], [2] gave two parametric solutions of the Diophantine equation

$$A^4 + B^4 = C^4 + D^4, \tag{1}$$

but the general solution is presently unknown. We give a geometric derivation of Euler's solutions and apply the geometric methods to obtain a new parametric solution, several new particular solutions and two complex parametric solutions. Related work by other investigators is also discussed. In a previous study [3], [4], the 46 smallest primitive solutions of (1) (those in which any factor common to A, B, C, D has been removed) were found by generating and sorting integers of the form $a^4 + b^4$ on a digital computer. These primitive solutions will be referred to as $\sigma_1, \sigma_2, \ldots,$ ranked in order of increasing magnitude of the common sum $A^4 + B^4$. We say $\sigma = (a, b, c, d)$ is a solution if $A = a$, $B = b$, $C = c$, $D = d$ satisfies (1). For example, $\sigma_1 = (158, 59, 134, 133)$, $\sigma_7 = (1203, 76, 1176, 653)$, $\sigma_{14} = (3494, 1623, 3351, 2338)$, $\sigma_{39} = (12231, 2903, 10381, 10203)$, and $\sigma_{46} = (15109, 581, 14723, 8461)$.

B. EULER'S ALGEBRAIC SOLUTION

Euler treated (1) by setting

$$A = p + q \qquad B = r - s \qquad C = p - q \qquad D = r + s \qquad \text{or} \tag{2.1}$$

$$2p = A + C \qquad 2q = A - C \qquad 2r = D + B \qquad 2s = D - B \tag{2.2}$$

which reduces (1) to

$$pq\,(p^2 + q^2) = rs\,(r^2 + s^2). \tag{3}$$

To each set of four positive integers A, B, C, D satisfying (1) there correspond 64 sets of integers (p, q, r, s) satisfying (3) obtained by using (2.2) and generating trivially different sets such as (p, q, s, r), (q, p, r, s), (p, -q, r, -s). A quite distinct solution of (3) is obtained by interchanging C and D in (2.1):

$$A = p_1 + q_1 \qquad B = r_1 - s_1 \qquad C = r_1 + s_1 \qquad D = p_1 - q_1 \tag{4.1}$$

$$2p_1 = A + D \qquad 2q_1 = A - D \qquad 2r_1 = C + B \qquad 2s_1 = C - B \tag{4.2}$$

and there are similarly 64 sets such as (p_1, q_1, r_1, s_1) which satisfy (3). For σ_1 we have $(p, q, r, s) = (146, 12, 96, 37)$ and $(2p_1, 2q_1, 2r_1, 2s_1) = (291, 25, 193, 75)$. The paired solutions to (3) are related by the equations

$$\begin{aligned} 2p_1 &= p + q + r + s & 2p &= p_1 + q_1 + r_1 + s_1 \\ 2q_1 &= p + q - r - s & 2q &= p_1 + q_1 - r_1 - s_1 \\ 2r_1 &= p - q + r - s & 2r &= p_1 - q_1 + r_1 - s_1 \\ 2s_1 &= p - q - r + s & 2s &= p_1 - q_1 - r_1 + s_1 . \end{aligned} \tag{5}$$

On substituting $z = q/s$, $v = (rs/pq) - 1$, (3) becomes

$$(z^2-1)^2 + v(3z^4-4z^2+1) + v^2(3z^4-6z^2) + v^3(z^4-4z^2) - v^4z^2 = [\frac{s}{p}(z^2-v-1)]^2 \tag{6}$$

and Euler made the quartic function of v on the left-hand side of (6) a square in two ways. First it was equated to $[(z^2-1) + fv + gv^2]^2$ and f and g were chosen to make the coefficients of v and v^2 in the resulting equation vanish. A rational solution of v as a function of z results. Euler [1] did not give the final algebraic result explicitly but did give the numerical solutions (2061283, 1584749, 2219449, 555617) and σ_{39} for z = 2 and z = 3, respectively. By completing the polynomial calculation, we find a solution of the 13th degree:

$$\begin{aligned} p &= 3(z^2-1)^2(9z^8-44z^6+190z^4+100z^2+1) \\ q &= z(z^{12}-214z^{10}-2481z^8-2804z^6-2481z^4-214z^2+1) \\ r &= 3z(z^2-1)^2(z^8+100z^6+190z^4-44z^2+9) \\ s &= z^{12}-214z^{10}-2481z^8-2804z^6-2481z^4-214z^2+1. \end{aligned} \tag{7}$$

These polynomials do not seem to have been previously recorded. In the second approach the quartic function in (6) was equated to $[(1+dv)(z^2-v-1)]^2$ and upon removing the common factor (z^2-v-1) there results

$$v^3(z^2+d^2) + v^2(3z^2-d^2z^2+d^2+2d) + v(3z^2-2dz^2+2d+1) = 0$$

which is solved rationally by choosing d so that the coefficient of v vanishes. This leads to the well known solution [2] of the 7th degree:

$$\begin{aligned} p &= 2(4z^6+z^4+10z^2+1) & r &= 2z(z^6+10z^4+z^2+4) \\ q &= -z(z^2+1)(z^4-18z^2+1) & s &= -(z^2+1)(z^4-18z^2+1) \end{aligned} \tag{8}$$

which produces σ_1, σ_7, σ_{14} for $z = 3, 2, 5$. In addition to the two parametric solutions, Euler [5] found the particular solution $\sigma_5 = (542, 103, 514, 359)$ by making special assumptions.

C. GEOMETRIC INTERPRETATION

If we set

$$x = p/s \qquad y = r/s \qquad z = q/s \tag{9}$$

then (3) becomes

$$y^3 + y = x^3 z + xz^3. \tag{10}$$

For a fixed rational z we may interpret (10) as the Cartesian equation of a cubic curve K in the xy plane, symmetric with respect to the origin and passing through the rational point P with coordinates $x = 1$, $y = z$. Each rational point on K corresponds to a solution of (3) through the relations $p = xs$, $q = zs$, $r = ys$ for arbitrary s. The point P gives the trivial solution $p = s$, $q = r$. Now a straight line in the xy plane intersects K in one or three real points (possibly at infinity) so the tangent to K at P_0 (x_0, y_0) intersects K again in P_1 (x_1, y_1). If P_0 is a rational point, the tangent has a rational equation, and as x_0, x_0, x_1 are then the three zeros of a rational cubic polynomial, P_1 is also a rational point on K.

The tangent at P_0 has slope $t = (z^3 + 3x_0^2 z)/(3y_0^2 + 1)$ and the coordinates of P_1 are $x_1 = -2x_0 - [3t^2 (y_0 - tx_0)/(t^3 - z)]$, $y_1 = t(x_1 - x_0) + y_0$. By setting $x_0 = 1$, $y_0 = z$ we find

$$x_1 = \frac{-2\,(4z^6 + z^4 + 10\,z^2 + 1)}{(z^2 + 1)\,(z^4 - 18\,z^2 + 1)} \qquad y_1 = \frac{-2z\,(z^6 + 10\,z^4 + z^2 + 4)}{(z^2 + 1)\,(z^4 - 18\,z^2 + 1)}$$

and if $s = -(z^2 + 1)\,(z^4 - 18\,z^2 + 1)$, the solution (8) results.

If $z = m^3$, K has the rational asymptote $y = mx$ and a line through P_0 (x_0, y_0)

parallel to this asymptote must intersect K again in a rational point $P_2(x_2, y_2)$. We find

$x_2 = [1 + (y_0 - mx_0)^2]/3m^2$, $y_2 = m(x_2 - x_0) + y_0$ and if $x_0 = 1$, $y_0 = z$ then $x_2 = (m^6 - 2m^4 + m^2 + 1)/3m^2$, $y_2 = m(m^6 + m^4 - 2m^2 + 1)/3m^2$.

Setting $s = 3m^2$ there results the following solution to (3):

$$p = m^6 - 2m^4 + m^2 + 1 \qquad r = m(m^6 + m^4 - 2m^2 + 1)$$
$$q = 3m^5 \qquad s = 3m^2 \tag{11}$$

This solution was given by A. Gérardin [6] and an algebraic derivation of it due to P. S. Dyer [7] can be found also in Hardy and Wright [8]. The solutions to (1) produced by (11) are in fact the same as those given by (8). The explanation is that the rational transformation $m = (z + 1)/(z - 1)$ carries (p, q, r, s) of (11) into polynomials proportional to $(-s_1, r_1, p_1, -q_1)$ obtained by applying the transformation (5) to (p, q, r, s) of (8). If we reverse this procedure for Euler's first solution, applying (5) to (7) and then substituting $z = (m + 1)/(m - 1)$, we get the following solution which is simpler than, but equivalent to, (7):

$$\begin{aligned}
p &= m(m^{12} + 2m^{10} - 3m^8 + 7m^6 - 12m^4 + 5m^2 + 1) \\
q &= m^{12} - 4m^{10} + 3m^8 + 4m^6 - 6m^4 - m^2 + 1 \\
r &= m(m^{12} - m^{10} - 6m^8 + 4m^6 + 3m^4 - 4m^2 + 1) \\
s &= m^{12} + 5m^{10} - 12m^8 + 7m^6 - 3m^4 + 2m^2 + 1 .
\end{aligned} \tag{12}$$

This solution can also be found geometrically using the fact that a line joining two rational points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ of K intersects K in a third rational point $P_3(x_3, y_3)$. The slope of the line is $t = (y_2 - y_1)/(x_2 - x_1)$ and $x_3 = -[x_1 + x_2 + 3t^2(y_1 - tx_1)/(t^3 - z)]$, $y_3 = t(x_3 - x_1) + y_1$. Given any initial solution (p_0, q_0, r_0, s_0) of (3), take $x_1 = 1$, $y_1 = z = q_0/s_0$, $x_2 = p_0/s_0$, $y_2 = r_0/s_0$. The slope t is then $(r_0 - q_0)/(p_0 - s_0)$. If we take (p_0, q_0, r_0, s_0) to be the polynomials (s, r, q, p) of (11), then $t = m$ and the new rational point (x_3, y_3) gives a solution equivalent to (12). A new solution of the 19th degree is obtained by taking $(p_0, q_0, r_0, s_0) = (r, s, q, p)$ from (11), and $x_1 = -1$, $y_1 = -z = -q_0/s_0$, $x_2 = p_0/s_0$, $y_2 = r_0/s_0$ which leads to

$$p = m^{19} - m^{18} - 3m^{17} - 3m^{16} + 21m^{15} - 9m^{14} - 44m^{13} + 74m^{12} - 39m^{11}$$
$$- 21m^{10} + 84m^{9} - 132m^{8} + 115m^{7} - 73m^{6} + 45m^{5} - 21m^{4} + 12m^{3} - 6m^{2} + m - 1$$
$$q = 3m^{2}\,(m^{12} - 4m^{10} + 18m^{9} - 36m^{8} + 27m^{7} + m^{6} - 9m^{5} + 9m^{4} - 9m^{3}$$
$$+ 5m^{2} + 1)$$
$$r = 3m^{2}\,(m^{15} - 3m^{14} - m^{13} + 11m^{12} - 12m^{11} + 4m^{10} + 10m^{9} - 30m^{8} + 39m^{7}$$
$$- 37m^{6} + 41m^{5} - 33m^{4} + 16m^{3} - 8m^{2} + 3m - 1)$$
$$s = (m^{6} - 2m^{4} + m^{2} + 1)\,(m^{12} - 4m^{10} + 18m^{9} - 36m^{8} + 27m^{7} + m^{6} - 9m^{5} \qquad (13)$$
$$+ 9m^{4} - 9m^{3} + 5m^{2} + 1).$$

These polynomials give σ_1 for $m = -1$ and new solutions such as (A, B, C, D) = (134413, 34313, 114613, 111637), (1057167, 552059, 1054067, 545991) for $m = 2, -2$. The particular result for $m = 2$ was found by E. Fauquembergue [9] using another algebraic method.

B. Segre [12] has given a general geometric treatment of fourth order Diophantine equations which correspond to quartic surfaces containing rational lines. In his paper two geometric transformations are introduced which for (1) lead to Euler's solution (8) and more generally to a discontinuous infinity of rational parametric solutions, not given explicitly.

D. COMPLEX SOLUTIONS

The rational complex points $x = iz$, $y = i$ lie on K and the tangent at each of these points has a second rational intersection with K, giving the following solutions to (3) of the 5th degree in Gaussian integers:

$$p = iz\,(z^4 - 2) \qquad r = i\,(-2z^4 + 1)$$
$$q = z\,(z^4 + 1) \qquad s = z^4 + 1 \qquad (14.1)$$

$$p = -iz\,(z^4 + 2) \qquad r = -i\,(2z^4 + 1)$$
$$q = z\,(z^4 - 1) \qquad s = z^4 - 1 \qquad (14.2)$$

E. PARTICULAR SOLUTIONS

By starting with any of the polynomial solutions already given and employing the geometric techniques discussed here, further parametric solutions may be obtained. Whether or not such procedures eventually yield all rational soultions to (1) is not apparent. The primitives $\sigma_2 = (239, 7, 227, 157)$ and $\sigma_3 = (292, 193, 257, 256)$ found by A. Werebrusow [10] are not produced by Euler's formulas (nor are any of the other computer-derived solutions except σ_1, σ_7, σ_{14}, σ_{39}), but are nevertheless geometrically derivable one from the other. By taking $z = 2/25$ in (10), the two rational points $x = 233/75$, $y = 82/75$ and $x = -274/225$, $y = -32/225$ on K correspond to σ_2, σ_3 and the line joining these points intersects K in the trivial point $x = -1$, $y = -2/25$. A number of similar relationships were found to hold among certain sets of the smallest known primitives, specifically $(\sigma_2, \sigma_3, \sigma_{11}, \sigma_{18})$, $(\sigma_4, \sigma_{13}, \sigma_{22})$, (σ_6, σ_{45}), $(\sigma_{16}, \sigma_{40})$, $(\sigma_{19}, \sigma_{41})$, and $(\sigma_{36}, \sigma_{46})$.

The geometric methods can be applied to particular numerical solutions and in some cases result in new solutions which do not involve overly large integers. Examples are (A, B, C, D) = (15265, 6101, 13085, 12743), (27407, 758, 27374, 7217), and (31731, 5468, 27661, 25596) derived from σ_{25}, σ_{38}, and σ_{15}, respectively.

F. ANOTHER SOLUTION

T. Hayashi [11] showed that every solution of Diophantine equation

$$3u^4 + v^4 = w^2 \tag{15}$$

leads to a solution to (1). We can express his result by stating that if u, v, w satisfy (15), then

$$p = 2u^3 (2u^4 + v^4) \qquad r = 2u^6 v$$

$$q = uv^4 w \qquad s = vw(2u^4 + v^4)$$

satisfy (3). Several systems are known which produce from one solution u_1, v_1, w_1 to (15) a new solution u_2, v_2, w_2; for example,

$$u_2 = 2u_1 v_1 w_1 \qquad v_2 = 2v_1^4 - w_1^2 \qquad w_2 = w_1^4 + 12u_1^4 v_1^4 .$$

The least solutions to (15) are (u, v, w) = (1, 1, 2), (2, 1, 7), and (3, 11, 122) which yield, respectively (A, B, C, D) = (2, 1, 1, 2), (542, 103, 514, 359) = σ_5, and (4970416, 1139811, 4962397, 1539492).

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) Aerospace Corporation El Segundo, California	2a REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP

3 REPORT TITLE

COLLECTED RESULTS ON NUMERICAL RESEARCH

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

5 AUTHOR(S) (Last name, first name, initial)

Lander, Leon J. and Parkin, Thomas R.

6 REPORT DATE	7a TOTAL NO OF PAGES	7b NO OF REFS
April 1967	90	56

8a CONTRACT OR GRANT NO AF 04(695)-1001	9a ORIGINATOR'S REPORT NUMBER(S) TR-1001(9990)-2
b. PROJECT NO	
c	9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d	SSD-TR-67-115

10 AVAILABILITY/LIMITATION NOTICES

This document has been approved for public release and sale; its distribution is unlimited.

11 SUPPLEMENTARY NOTES	12 SPONSORING MILITARY ACTIVITY Space Systems Division Air Force Systems Command Los Angeles, California

13 ABSTRACT

A compilation of results from research in number theory involving the use of a digital computer is presented. The research is related to characteristics of prime numbers, equal sums of powers of integers, differences of powers of integers, equal sums of fifth powers, and the classic problem of finding two equal sums of two biquadrates.

UNCLASSIFIED

Security Classification

KEY WORDS

Number theory
Numerical research
Characteristics of prime numbers
Equal sums of powers of integers
Differences of powers of integers
Equal sums of fifth powers
Two equal sums of two biquadrates
Biquadrates
Euler's Diophantine equation
Diophantine equation

Abstract (Continued)

UNCLASSIFIED

Security Classification